

# RECOGNIZING NULLHOMOTOPIC MAPS INTO THE CLASSIFYING SPACE OF A KAC–MOODY GROUP

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**ABSTRACT.** This paper extends certain characterizations of nullhomotopic maps between  $p$ -compact groups to maps with target the  $p$ -completed classifying space of a connected Kac–Moody group and source the classifying space of either a  $p$ -compact group or a connected Kac–Moody group. A well known inductive principle for  $p$ -compact groups is applied to obtain general, mapping space level results. An arithmetic fiber square computation shows that a null map from the classifying space of a connected compact Lie group to the classifying space of a connected topological Kac–Moody group can be detected by restricting to the maximal torus. Null maps between the classifying spaces of connected topological Kac–Moody groups cannot, in general, be detected by restricting to the maximal torus due to the nonvanishing of an explicit abelian group of obstructions described here. Nevertheless, partial results are obtained via the application of algebraic discrete Morse theory to higher derived limit calculations which show that such detection is possible in many cases of interest.

## 1. INTRODUCTION

After the proof of Sullivan conjecture by Carlson, Lannes and Miller [9, 36, 38], determining the homotopy type of the mapping space  $\text{map}(BG', BG)$  for  $G'$  and  $G$  compact Lie groups became much more technically accessible. Among the first and most broadly applicable results where characterizations of nullhomotopic maps by Friedlander–Mislin [20], Zabrodsky [49], and Jackowski–McClure–Oliver [27]. Later, Møller [40] applied an inductive principle [16, 9.2] to obtain characterizations of nullhomotopic maps between  $p$ -compact groups. Kac–Moody groups provide a generalization of compact Lie groups and characterizations of null maps between the classifying spaces of rank two (and non-affine) Kac–Moody groups have been given by Aguadé–Ruiz [2].

This article adapts the inductive arguments of [16, 40] to recognize null maps between Kac–Moody group classifying spaces of all ranks. Throughout the term Kac–Moody group refers to the unitary form associated to some Kac–Moody Lie algebra [35, 7.4] (consistent with [1, 2, 5, 33]). We refer the reader to [35] for a thorough account of Kac–Moody groups and [33] for an introduction to their topology. See Appendix A for a discussion of how our results apply to Kac–Moody groups associated to derived Kac–Moody Lie algebras.

All the previous work mentioned above relies on  $p$ -local techniques. Here we obtain the following characterization of nullhomotopic maps with target the  $p$ -completed classifying space of a Kac–Moody group.

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**Theorem A.** *If  $K$  is a Kac–Moody group and  $\Gamma$  is a connected  $p$ -compact or Kac–Moody group with maximal torus  $\iota: T \rightarrow \Gamma$ , then the null component of the unpointed mapping space  $\text{map}(B\Gamma, BK_p^\wedge)_0$  is canonically equivalent to  $BK_p^\wedge$  and  $f: B\Gamma \rightarrow BK_p^\wedge$  is nullhomotopic if and only if  $f|_{BT}$  is nullhomotopic.*

Under the more restrictive hypothesis that  $\Gamma$  and  $K$  are rank two (and non-affine) Kac–Moody groups, Theorem A is shown in [2] but not stated explicitly; see [27, Theorem 3.11] for a compact Lie group statement.

The problem of recognizing when integral maps  $B\Gamma \xrightarrow{g} BK$ , where now  $\Gamma$  is a connected compact Lie or Kac–Moody group, are nullhomotopic exposes new subtleties. For the cases studied in [2, 27], it is proved therein that the homotopy class of  $g$  is detected by the collection of homotopy classes of its  $p$ -completions  $g_p^\wedge$  for all primes  $p$ . Hence, Theorem A implies that null  $g$  are detected by restricting to the maximal torus. In our more general setting, we prove that there exist homotopically nontrivial  $B\Gamma \xrightarrow{g} BK$  such that all  $g_p^\wedge$  and the rationalization  $g_\mathbb{Q}^\wedge$  are null, so Theorem A alone cannot recognize null homotopic  $g$ .

We show that  $g|_{BT}$  is null still implies that  $g_\mathbb{Q}^\wedge$  is null (see Sections 5–6) and:

**Theorem B.** *If  $K$  is a Kac–Moody group and  $\Gamma = G$  is a connected compact Lie group, then  $g: BG \rightarrow BK$  is nullhomotopic if and only if  $g|_{BT}$  is nullhomotopic.*

However—if  $\Gamma$  is not compact—then explicit obstructions parameterize the homotopy classes of  $g$  such that  $g|_{BT}$  is null (see Theorem 5.3). These obstructions are frequently nonzero, e.g. if  $\Gamma = K$  is affine or  $K$  is compact Lie but  $\Gamma$  is not (see Examples 5.4 and 7.1, respectively).

These negative results may disappoint readers, but they do not preclude a Kac–Moody analog of the description given in [27, Theorems 2 and 3] of  $\text{map}(BG, BG)$  for  $G$  a compact *simple* Lie group. Because the standard representation of the Weyl group of an affine Kac–Moody group is not irreducible, affine  $K$  are not natural analogs of simple  $G$ . In contrast, we prove a partial step toward such a result.

**Theorem C.** *If  $K$  is an indecomposable, non-affine and 2-spherical Kac–Moody group, then  $g: BK \rightarrow BK$  is nullhomotopic if and only if  $g|_{BT}$  is nullhomotopic.*

The restriction to indecomposable  $K$  avoids direct product decompositions while the 2-spherical condition (see Definition 6.4) implies a nilpotency hypothesis on  $H^*(BK, \mathbb{Q})$  that is necessary for the conclusion that  $g$  is nullhomotopic if and only if  $g|_{BT}$  is nullhomotopic. In particular, we verify this nilpotency hypothesis by direct computation and give examples of  $H^*(BK, \mathbb{Q})$  as a ring which have not appeared previously (see Examples 7.1 and 7.3), but we do not know of any of example of an indecomposable, non-affine Kac–Moody group for which this nilpotency hypothesis fails (see Question 8.7). These computations are facilitated—technically and conceptually—by the application of algebraic discrete Morse theory [11, 28, 44]. Theorem C can be interpreted as a uniqueness result for the trivial unstable Adams operation (cf. [27, Theorem 1] and [2, § 10]). Though unstable Adams operations for Kac–Moody groups have been constructed, their homotopical uniqueness is not settled for rank greater than 2 [17, Theorem D and Question 4.4]).

Given the success found by studying  $p$ -compact groups and extending Lie theory to Kac–Moody theory (see, e.g., [23, 39] and [35], resp.), it is hoped that aspects of  $p$ -compact group theory will extend to a “homotopy version of a Kac–Moody group” at  $p$  called a homotopy Kac–Moody group. This paper is uncommonly successful in repurposing  $p$ -compact group methodology (cf. [1, 2, 32, 33]), especially inductive techniques; see Theorem 4.1 for a generalization of Theorem A to this perspective. For instance, the class of spaces to which our  $p$ -local results apply includes all representatives of the homotopy types  $\mathcal{S}^*$  introduced in [1]. Elements

of  $\mathcal{S}^*$  are constructed as pushouts of  $p$ -compact groups and generalize  $p$ -completed, rank two, Kac–Moody group classifying spaces. Indeed, our structure theorems are put into practice in a forthcoming paper [18] where maps between representatives of  $\mathcal{S}^*$  are studied.

While the analogy to the compact Lie case is strong at  $p$ , this paper gives explicit evidence that the classifying spaces of Kac–Moody groups are rationally less analogous to the classifying spaces of compact Lie group. It has already been observed that constructing integral mapping spaces via the arithmetic fibre square results in new issues—even if  $BK$  is rationally an Eilenberg–MacLane space (cf. [2, Remark 3.4, Theorem 6.2]). We believe that the problems of describing the rational homotopy type of and constructing integral mapping spaces between the classifying spaces of Kac–Moody groups deserve investigation apart from  $p$ -local concerns. We hope that our more preliminary rational and integral results stimulate further research in these directions. Our  $H^*(BK, \mathbb{Q})$  computations may find special independent interest and efforts have been made so that they can be read somewhat independently.

*Conventions, etc.* Since the author wishes to highlight the reuse of  $p$ -compact group methods, this paper does not attempt to be completely self-contained and points of comparison are documented throughout. We follow the (mostly) standard definitions and notations for  $p$ -compact groups appearing in [39]. Notably, we follow refer to  $BY$  as a  $p$ -compact group rather than the classifying space of a  $p$ -compact group  $Y := \Omega BY$ . For the reader’s convenience, we maintain the following notational conventions:

- $K$  = a (minimal, split, unitary) Kac–Moody group [33], [35, 7.4]
- $G$  = a compact Lie group
- $\Gamma$  = a connected compact Lie or Kac–Moody group
- $\text{rank}(\Gamma)$  = the rank of a maximal torus of  $\Gamma$
- $BY$  = a  $p$ -compact group [39, 1.10]
- $BP$  = a  $p$ -compact toral group [39, 2.1] or, more briefly, a  $p$ -toral group
- $\text{cd}_{\mathbb{F}_p} Y$  = the maximal nonzero dimension of  $H^*(Y, \mathbb{F}_p)$
- $(-)_{\mathbb{F}_p}$  = the  $\mathbb{F}_p$ -homology localization functor [13]
- $(-)_{\wedge_p}$  = the Bousfield–Kan  $p$ -completion (homotopy) functor [4]
- $(-)_{\hat{\mathbb{Q}}}$  = the Bousfield–Kan rationalization (homotopy) functor [4]
- $BX$  = an analog of  $BK_{\wedge_p}$  with associated  $p$ -compact groups  $BX_I$

and recommend having [16, 40] at hand while reading Section 2. Whenever a space level  $p$ -completion is functor needed, we will be able to apply  $(-)_{\mathbb{F}_p}$  which, for the spaces under consideration, will coincide with  $(-)_{\wedge_p}$ , up to homotopy, (see, e.g., (3) below.)

The author can also recommend subsets of the article which may appeal to various readers. Sections 2–4 prove Theorem A and constitute the  $p$ -local, homotopy Kac–Moody group story. Sections 5–8 can be read by taking Theorem A as a starting point and place Theorem C in context with proof. Section 5 outlines the local to global construction of integral  $B\Gamma \xrightarrow{g} BK$  including a proof of Theorem B. It can be read alone or supplemented by Section 6 which gives further detail on vanishing and homotopical uniqueness. Section 7 displays examples of  $H^*(BK, \mathbb{Q})$  and with reference to definitions and perhaps some of Section 6 for context, can be read as introduction to a computational approach to  $H^*(BK, \mathbb{Q})$  via algebraic discrete Morse theory. Such a reading of Section 7 can also be supplemented by Section 8 where the key vanishing result (Theorem 6.9) for Theorem C is proven by inputting invariant theory into this approach.

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## 2. STRUCTURE RESULTS BY INDUCTION

This section extends some structural, mapping space level results ([16, 10.1] and [40, 2.11, 6.1, 6.6, 6.7]) concerning maps between  $p$ -compact groups to maps from a  $p$ -compact group  $BY$  into a space  $BX$  within a class of potential (connected) homotopy Kac–Moody groups. Specifically, the spaces in this class are constructed as follows

$$(1) \quad BX := (\operatorname{hocolim}_{I \in \mathbf{S}} BX_I)_p^\wedge = (\operatorname{hocolim}_{\mathbf{S}} D)_p^\wedge$$

where  $D: \mathbf{S} \rightarrow p\text{-}\mathbf{cpt}$  is some diagram of monomorphisms of connected  $p$ -compact groups (i.e.  $BX_I = D(I)$  is simply-connected) such that the index category  $\mathbf{S}$  is a finite, contractible poset.

Let us first recall how a  $p$ -completed Kac–Moody group classifying space can be constructed as in (1). See Section 4 for further examples. For any Kac–Moody group  $K$ , we have a homotopy decomposition

$$(2) \quad \operatorname{hocolim}_{I \in \mathbf{S}} BK_I \xrightarrow{\sim} BK$$

where  $K_I \leq K$  are connected Lie groups and  $\mathbf{S}$  is the finite poset of spherical subsets of  $\{1, \dots, \operatorname{size}(A)\}$  ordered by inclusion [33] for  $A$  the associated generalized Cartan matrix. Let  $D_K$  denote this diagram  $D_K: \mathbf{S} \rightarrow \mathbf{Spaces}$  given by  $I \mapsto BK_I$ . In particular,  $\emptyset \in \mathbf{S}$  is an initial object so that  $|\mathbf{S}|$  is contractible and  $BK_\emptyset = BT$  is the classifying space of the maximal torus of  $K$ . Moreover, each  $I \leq J$  in  $\mathbf{S}$  corresponds to an inclusion of subgroups with each  $\emptyset \leq J$  corresponding to the standard maximal torus inclusion  $BT \xrightarrow{L_I} BK_I$ . Since all spaces involved are simply-connected,  $p$ -completion coincides, up to homotopy, with  $\mathbb{F}_p$ -homology localization so that we have weak equivalences

$$(3) \quad [\operatorname{hocolim}_{I \in \mathbf{S}} (BK_I)_{\mathbb{F}_p}]_{\mathbb{F}_p} \xleftarrow{\sim} [\operatorname{hocolim}_{I \in \mathbf{S}} BK_I]_{\mathbb{F}_p} \xrightarrow{\sim} BK_{\mathbb{F}_p} \simeq BK_p^\wedge.$$

In particular,  $\operatorname{hocolim}_{I \in \mathbf{S}} (BK_I)_{\mathbb{F}_p}$  is simply-connected by Seifert–van Kampen theory (see, e.g., [14]) and any  $BX$  as in (1) is simply-connected,  $p$ -complete and  $\mathbb{F}_p$ -local for the same reasons.

The key point is to apply an inductive principle for  $p$ -compact groups:

**Theorem 2.1** ([16, 9.2]). *The only class  $\mathcal{C}$  of  $p$ -compact groups  $BY$  with the five properties:*

- (i)  $\mathcal{C}$  is closed under homotopy equivalence,
- (ii)  $\mathcal{C}$  contains the point,
- (iii)  $BY_0 \in \mathcal{C}$  implies  $BY \in \mathcal{C}$ ,
- (iv) for simply-connected  $BY$ ,  $B(Y/Z) \in \mathcal{C}$  implies  $BY \in \mathcal{C}$ , and
- (v) for simply-connected  $BY \simeq B(Y/Z)$ , if  $\operatorname{cd}_{\mathbb{F}_p} Y' < \operatorname{cd}_{\mathbb{F}_p} Y$  implies  $BY' \in \mathcal{C}$ , then  $BY \in \mathcal{C}$ .

*is the class of all  $p$ -compact groups.*

In the above,  $BY_0$  is the homotopy fiber of the canonical projection  $BY \rightarrow B\pi_1(BY)$  and  $BZ \rightarrow BY \rightarrow B(Y/Z)$  is also a homotopy fiber sequence of  $p$ -compact groups where  $BZ$  is the identity component of  $\operatorname{map}(BY, BY)$ .

Inductive arguments will be lifted from [16, §9,10] and [40, §2,6] as outlined below to reduce our results to the situation that the source  $p$ -compact group  $BY$  is the classifying space of  $p$ -toral group. Proofs for a  $p$ -toral source are given in Section 3. See Remark 3.3 for a discussion of alternative approaches.

Let us now state our generalization of [16, 10.1].

**Theorem 2.2.** *For  $BX$  as in (1) the natural map*

$$(4) \quad \text{map}(*, BX) \longrightarrow \text{map}(BY, BX)_0$$

*is a weak equivalence for any  $p$ -compact group  $BY$ .*

As in [16], Theorem 2.1 and the following lemma can be applied to show that it is sufficient to prove Theorem 2.2 for  $BY$  is the classifying space of  $p$ -toral group.

**Lemma 2.3.** *Let  $X$  be a  $\mathbb{F}_p$ -local space such that the canonical map*

$$(5) \quad \text{map}(*, X) \longrightarrow \text{map}(BY, X)_0$$

*is a weak homotopy equivalence for all  $BY$  a  $p$ -toral  $p$ -compact group. Then (5) is a weak equivalence for any  $p$ -compact group  $BY$ .*

**Proof of Lemma 2.3:** Following [16, §9,10], it enough to show that the class of  $p$ -compact groups for which (5) is a weak equivalence has the five properties of Theorem 2.1. Clearly, (i) and (ii) hold.

Let us now check (iii). By our inductive assumption, the canonical

$$(6) \quad \text{map}(*, X) \longrightarrow \text{map}(BY_0, X)_0$$

is a weak homotopy equivalence. Setting  $\pi := \pi_1(BY)$  and noting  $BY \simeq (BY_0)_{h\pi}$ , we obtain a weak equivalence

$$\text{map}(B\pi, X) \longrightarrow \coprod_{\phi|BY_0 \simeq 0} \text{map}(BY, X)_\phi$$

which restricts to a weak equivalence of components

$$(7) \quad \text{map}(B\pi, X)_0 \longrightarrow \text{map}(BY, X)_0.$$

By hypothesis, (5) with  $BY$  replaced by  $B\pi$ —a  $p$ -toral group—is also a weak equivalence. Composing with (7) yields that  $BY \rightarrow B\pi \rightarrow *$  induces the desired weak equivalence.

A similar argument reduces (iv) to the hypothesis that (5) is a weak equivalence for  $BY = BZ$ . The observation that  $BZ$  is a  $p$ -toral  $p$ -compact group [16] completes the check of (iv).

To check (v), use the homology decomposition for  $BY$

$$(8) \quad \text{hocolim}_{(V, BV \xrightarrow{v} BY) \in \mathbf{A}^{op}} \text{map}(BV, BY)_v \longrightarrow BY.$$

where objects of  $\mathbf{A}^{op}$  consist of pairs  $(V, BV \xrightarrow{v} BY)$  for  $V$  a finite elementary abelian  $p$ -group [16]. Now we have weak equivalences

$$\begin{aligned} \text{map}(*, X) &\rightarrow \text{map}(|\mathbf{A}^{op}|, X) \rightarrow \text{holim}_{\mathbf{A}} \text{map}(*, X) \\ &\rightarrow \text{holim}_{\mathbf{A}} \text{map}(\text{map}(BV, BY)_f, X)_0 \end{aligned}$$

using the fact that the  $\mathbb{F}_p$ -localization of  $|\mathbf{A}^{op}|$  is contractible (cf. [16, 40]).  $\square$

By design, the hypotheses of Lemma 2.3 are compatible with Theorem 2.2.

**Reduction of the proof of Theorem 2.2:** Recalling that  $BX$  is  $\mathbb{F}_p$ -local, we only need to show that (4) is a weak equivalence for  $p$ -toral  $BY := BP$  by Lemma 2.3.  $\square$

Given Theorem 2.2, the proof of Lemma 2.11 in [40] is now directly applicable.

**Theorem 2.4.** *For  $BX$  as in (1) and any homotopy fiber sequence of  $p$ -compact groups*

$$(9) \quad B\overline{Y} \rightarrow B\tilde{Y} \xrightarrow{f} BY$$

the natural map induced by  $f$  factors as

$$(10) \quad \begin{array}{ccc} & \coprod_{\phi|_{B\bar{Y} \simeq 0}} \text{map}(B\tilde{Y}, BX)_\phi & \\ \nearrow \sim & \downarrow & \\ \text{map}(BY, BX) & \xrightarrow{f^*} & \text{map}(B\tilde{Y}, BX) \end{array}$$

so that  $\text{map}(BY, BX)$  is identified, up to homotopy, as the space of maps  $B\tilde{Y} \xrightarrow{\phi} BY$  that restrict to nullhomotopic maps from  $B\bar{Y}$ .

**Proof:** By Theorem 2.2, the natural map

$$(11) \quad \text{map}(*, BX) \longrightarrow \text{map}(B\bar{Y}, BX)_0$$

is a weak equivalence. As in [40, §2], we may identify  $f^*$  in (10) with natural map

$$\text{map}(BY, \text{map}(*, BX)) \longrightarrow \text{map}(BY, \text{map}(B\bar{Y}, BX)_0).$$

□

With Theorem 2.4 in hand, it is now straightforward to adapt [40, §6].

**Theorem 2.5** (cf. [40, 6.1], [20, 3.3]). *Let  $BX$  be as in (1) and  $BY$  be a  $p$ -compact group. A map  $BY \xrightarrow{f} BX$  is nullhomotopic if and only if for all maps  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY$  the composition  $fe$  is nullhomotopic.*

**Proof:** Only the “if” direction requires proof, so we assume all compositions  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY \xrightarrow{f} BX$  are nullhomotopic. Following [40, §6], it enough to show that the class  $\mathbf{C}$  of  $p$ -compact groups such that the statement holds has the five properties of Theorem 2.1. The basic observation, which resulted in the present paper, is that arguments to be adapted are not too sensitive to the target of  $f$  (see (12-13) below). Clearly, (i) and (ii) hold.

Consider property (iii). By assumption, all compositions

$$B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY_0 \rightarrow BY \xrightarrow{f} BX$$

are nullhomotopic and thus  $BY_0 \rightarrow BY \xrightarrow{f} BX$  is nullhomotopic. Setting  $\pi := \pi_1(BY)$ , Theorem 2.4 implies that each  $f$  factors through  $B\pi$ , up to homotopy. Just as in [40, Proof of 6.3], we have a lift

$$(12) \quad \begin{array}{ccc} B\mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\hat{e}} BY & \xrightarrow{f} & BX \\ \text{mod } p^n \downarrow & \downarrow & \nearrow \bar{f} \\ B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} B\pi & & \end{array}$$

for any  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY$  as indicated. Thus, checking property (iii) is reduced to checking that  $B\pi \in \mathbf{C}$ .

Considering property (iv) and setting  $BZ$  to be the identity component  $\text{map}(BY, BY)_{id}$ , Theorem 2.4 implies that each  $f$  factors through  $BZ$ , up to homotopy. Just as in [40, Proof of 6.4], we have a lift

$$(13) \quad \begin{array}{ccc} B\mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\hat{e}} BY & \xrightarrow{f} & BX \\ \text{mod } p^n \downarrow & \downarrow & \nearrow \bar{f} \\ B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BZ & & \end{array}$$

for any  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY$  as indicated. Thus, the check of property (iv) is reduced to checking that  $BZ \in \mathbf{C}$ .

Recall that  $BZ$  and  $B\pi$  are always  $p$ -toral groups. Hence, it is sufficient to show that  $p$ -toral groups  $BP$  are in  $\mathbf{C}$  to finish the verification of properties (iii) and (iv). See Lemma 3.2 below.

For property (v), use the homology decomposition (8) for  $BY$ . In particular,  $BY$  being simply-connected and centerless implies each

$$\mathrm{cd}_{\mathbb{F}_p}(\mathrm{map}(BV, BY)_v) < \mathrm{cd}_{\mathbb{F}_p}(BY).$$

By induction, each  $\mathrm{map}(BV, BY)_v \xrightarrow{v} BY \xrightarrow{f} BX$  is nullhomotopic. The subspace of  $\mathrm{map}(BY, BX)$  consisting of all possible  $g$  such that all compositions  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BY \xrightarrow{g} BX$  are nullhomotopic is weakly equivalent to the subspace

$$\mathrm{holim}_{\mathbf{A}} \mathrm{map}(\mathrm{map}(BV, BY)_v, BX)_0.$$

To see that this space is path-connected (as desired) apply Theorem 2.2. In particular,  $\mathrm{holim}_{\mathbf{A}} BX$  is weakly equivalent to  $BX$  since the  $\mathbb{F}_p$ -localization of  $|\mathbf{A}^{op}|$  is contractible and  $BX$  is  $\mathbb{F}_p$ -local.  $\square$

Finally, we apply Theorem 2.5 as in [40, §6].

**Theorem 2.6** (cf. [40, 6.6, 6.7]). *Let  $BX$  be as in (1) and  $BY$  be a  $p$ -compact group with maximal torus  $BT \xrightarrow{l} BY$  and maximal torus normalizer  $BN \rightarrow BY$ . For any map  $BY \xrightarrow{f} BX$   $f|_{BN}$  is nullhomotopic if and only if  $f$  is nullhomotopic. If—in addition— $BY$  is simply-connected, then  $f|_{BT}$  is nullhomotopic if and only if  $f$  is nullhomotopic.*

**Proof:** See the proofs of Corollaries 6.6 and 6.7 in [40].  $\square$

### 3. MAPS FROM $p$ -COMPACT TORAL GROUPS

We begin our study of maps from  $p$ -toral groups  $BP$  into  $BX$  with the situation where  $P$  is finite. The following is a specialization of a result of Broto, Levi, and Oliver [6, Proposition 4.2]. Compare [5, Theorem 6.11].

**Proposition 3.1.** *Let  $D: \mathbf{S} \rightarrow p\text{-cpt}$  be a diagram of  $p$ -compact groups indexed over a finite poset  $\mathbf{S}$ . For any finite  $p$ -group  $\pi$  the canonical map*

$$(14) \quad [\mathrm{hocolim}_{I \in \mathbf{S}} \mathrm{map}(B\pi, BX_I)]_p^\wedge \xrightarrow{\sim} \mathrm{map}(B\pi, (\mathrm{hocolim}_{I \in \mathbf{S}} BX_I)_p^\wedge)$$

*is a weak homotopy equivalence.*

**Proof:** To apply [6, Proposition 4.2], we must check that  $\mathrm{map}(B\pi, BX_I)$  is  $p$ -complete and that certain higher limits over  $\mathbf{S}$  vanish in degrees greater than some uniform  $n$  depending on  $\mathbf{S}$ . See, for example, [39, Theorem 5.1] for the former condition. For the latter, recall [21, Appendix II.3], [47, 8.3] that all higher limits over  $\mathbf{S}$  in degrees greater than the maximum chain length of  $\mathbf{S}$  vanish.  $\square$

We now apply Proposition 3.1 to complete the proof of Theorem 2.2.

**Proof of Theorem 2.2:** We must still verify that (4) is a weak equivalence for  $BP := BY$  a  $p$ -toral group. The existence of a homotopy colimit approximation  $\mathrm{hocolim}_{n \rightarrow \infty} BP_n \rightarrow BP$  [39, 2.1] reduces the problem to showing

$$(15) \quad \mathrm{map}(*, BX) \longrightarrow \mathrm{map}(BP_n, BX)_0$$

is a weak equivalence. Now by Proposition 3.1,  $\mathrm{map}(BP_n, BX)_0$  is some component of  $\mathrm{hocolim}_{I \in \mathbf{S}} \mathrm{map}(BP_n, BX_I)$ . In particular,

$$\mathrm{map}(*, BX_I) \longrightarrow \mathrm{map}(BP_n, BX_I)_0$$

is a weak equivalence for each  $I \in \mathbf{S}$  by [16, 9.2]. For all  $J \leq I$  in  $\mathbf{S}$ , the space  $X_I/X_J$  in the homotopy fiber sequence

$$(16) \quad X_I/X_J \longrightarrow BX_J \longrightarrow BX_I$$

is  $\mathbb{F}_p$ -finite since  $BX_J \rightarrow BX_I$  is a monomorphism. In the induced homotopy fiber sequence,

$$\mathrm{map}(BP_n, X_I/X_J) \longrightarrow \mathrm{map}(BP_n, BX_J) \longrightarrow \mathrm{map}(BP_n, BX_I)$$

the Sullivan conjecture [38] gives that  $\mathrm{map}(BP_n, X_I/X_J) \simeq X_I/X_J$ . Since by construction  $\mathrm{map}(BP_n, BX_I)_0 \simeq BX_I$  is simply-connected, the associated homotopy fiber sequence over  $\mathrm{map}(BP_n, BX_I)_0$  implies that the set of components of  $\mathrm{map}(BP_n, BX_J)$  that push forward into  $\mathrm{map}(BP_n, BX_I)_0$  is isomorphic  $\pi_0(X_I/X_J)$  and (16) implies  $\pi_0(X_I/X_J) \cong \pi_0(BX_J) \cong \{*\}$ . Hence, we have a commuting square of weak equivalences

$$(17) \quad \begin{array}{ccc} \mathrm{hocolim}_{I \in \mathbf{S}} \mathrm{map}(*, BX_I) & \longrightarrow & \mathrm{hocolim}_{I \in \mathbf{S}} \mathrm{map}(BP_n, BX_I)_0 \\ \downarrow & & \downarrow (14) \\ \mathrm{map}(*, BX) & \longrightarrow & \mathrm{map}(BP_n, BX)_0 \end{array}$$

so that (15) is a weak equivalence for all  $n$  as desired.  $\square$

**Lemma 3.2.** *Theorem 2.5 holds under the additional hypothesis that  $BY$  is  $p$ -toral.*

**Proof:** Set  $BP := BY$  and fix an approximation  $\mathrm{hocolim}_{n \rightarrow \infty} BP_n \rightarrow BP$ . For each  $BP_n \xrightarrow{g} BX$  and any  $B\mathbb{Z}/p^k\mathbb{Z} \xrightarrow{e_n} BP_n$  there is a homotopy commutative diagram

$$\begin{array}{ccc} & & BX_I \\ & \nearrow \bar{g} & \downarrow \\ B\mathbb{Z}/p^k\mathbb{Z} & \xrightarrow{e_n} BP_n & \xrightarrow{g} BX \end{array}$$

for some  $I \in \mathbf{S}$  by the identification (14) for  $\pi = P_n$ . If we assume that all compositions  $ge_n$  are null, then any  $\bar{g}$  is null by applying [40, Lemma 6.2] to  $BX_I$ .

Moreover, the identification (17) implies that

- $g$  is null if and only  $\bar{g}$  is null and
- $BP \xrightarrow{f} BX$  is null if and only if  $f_n := f|_{BP_n}$  is null for all  $n \in \mathbb{N}$

as the uniqueness obstructions to extending the collection  $\{f_n\}_{n \in \mathbb{N}}$  to  $BP$  lie in  $\lim^1(\pi_1(\mathrm{map}(BP_n, BX)_{f_n}))$ . Hence, if  $B\mathbb{Z}/p^k\mathbb{Z} \xrightarrow{e} BP \xrightarrow{f} BX$  is null for all  $B\mathbb{Z}/p^k\mathbb{Z} \xrightarrow{e} BP$ , then  $f$  is null by taking  $g = f|_{BP_n}$  for varying  $n$ .  $\square$

**Remark 3.3.** As pointed out to the author by Castellana, it is also possible to adapt the arguments of Notbohm [41] or to use the subgroup decomposition for a  $p$ -compact group [10]—or even the subgroup decomposition for a  $p$ -local compact group [7, 4.6]—to show Theorem 2.2. In these and our approaches, the natural equivalence  $\mathrm{map}(B\mathbb{Z}/p\mathbb{Z}, BX)_0 \simeq BX$  induced by evaluation—obtained via (14)—and the fact that  $BX$  is  $p$ -complete and hence  $\mathbb{F}_p$ -local—obtained via Seifert–van Kampen theory—are essential for beginning the inductive process.  $\square$

#### 4. NULL MAPS BETWEEN HOMOTOPY KAC–MOODY GROUPS

Let us now detail some more specific applications. In addition to  $BK_p^\wedge$ , the results of the previous two sections apply to the  $p$ -completed classifying spaces of

- parabolic subgroups  $K_J$  (cf. [5]) with  $J$  not necessarily spherical or
- central quotients (cf. [1]) and further subgroups of the above (see [18]) and
- representatives of  $\mathcal{S}^*$  (as defined in [1]) as well as
- $p$ -completed Davis–Januszkiewicz spaces [42, 43]



That is, all these examples can be constructed as in (1) with the  $p$ -completed classifying space examples following (3), representatives of  $\mathcal{S}^*$  constructed as homotopy pushouts of  $p$ -completed compact Lie group classifying spaces, and Davis–Januszkiewicz spaces constructed as a finite poset diagram of torus classifying spaces with an initial object corresponding to the base point.

The following result indicates that the existence of a natural candidate for a maximal torus further simplifies the characterization of null homotopic maps.

**Theorem 4.1.** *For  $BX$  and  $BX'$  as in (1), the natural*

$$\mathrm{map}(*, BX') \longrightarrow \mathrm{map}(BX, BX')_0$$

*is a weak equivalence and  $BX \xrightarrow{f} BX'$  is nullhomotopic if and only if for all maps  $B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BX$  the composition  $fe$  is nullhomotopic. If—in addition—the diagram  $D$  defining  $BX$  has commuting squares*

$$(18) \quad \begin{array}{ccc} BT & \xlongequal{\quad} & BT \\ \downarrow \iota_I & & \downarrow \iota_J \\ BX_I & \xrightarrow{D(I \leq J)} & BX_J \end{array}$$

*for all  $I \leq J \in \mathbf{S}$  where  $BT \xrightarrow{\iota_I} BX_I$  are maximal tori, then we have a natural  $BT \xrightarrow{\iota} BX$  and  $f|_{BT}$  is nullhomotopic if and only if  $f$  is nullhomotopic.*

**Proof:** To obtain the natural weak equivalence, we note that Theorem 2.2 and the contractibility of  $|\mathbf{S}|$  imply that we have a commuting square of weak equivalences

$$(19) \quad \begin{array}{ccc} \mathrm{map}(*, BX') & \xrightarrow{\quad} & \mathrm{map}(BX, BX')_0 \\ \downarrow & & \downarrow j_I^* \\ \mathrm{map}(|\mathbf{S}|, BX') & \xrightarrow{\quad} & \mathrm{holim}_{I \in \mathbf{S}^{op}} \mathrm{map}(BX_I, BX')_0. \end{array}$$

For the first characterization of null maps, only the “if” direction requires proof and we assume that all compositions

$$B\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{e} BX_I \xrightarrow{j_I} BX \xrightarrow{f} BX'$$

where  $e$  is arbitrary and  $j_I$  is canonical are null. Theorem 2.5 implies that each  $fj_I$  is null so that (19) implies that  $f$  is null.

For the second characterization of null maps, only the “only if” direction requires proof and we assume that  $f|_{BT}$  is nullhomotopic. Recalling that all  $BX_I$  are assumed to be simply-connected, Theorem 2.6 implies that each  $f|_{BX_I}$  is null so that (19) implies that  $f$  is null.  $\square$

In rough terms, requiring (18) means that all  $BX_I$  for  $I \in \mathbf{S}$  have a common torus  $BT$  which can be added to the diagram  $D$  defining  $BX$  as an initial space and—since  $|\mathbf{S}|$  is contractible—the homotopy colimit of this extended diagram is still  $BX$ .

**Proof of Theorem A:** Any  $p$ -completed Kac–Moody group classifying space can be constructed as in (3) so that (18) holds for  $BT = (BK_\emptyset)_p^\wedge$ .  $\square$

The first three types of examples mentioned at the start of the section have (18). For instance, the pushouts constructing  $\mathcal{S}^*$  have a rank two torus as their initial spaces and the two associated are maps maximal tori. Of course, Davis–Januszkiewicz spaces that are not already the classifying space of a torus do not have (18). We note for completeness that Anjos and Granja [3] constructed a natural space as a homotopy pushout of connected Lie group classifying spaces that does not satisfy (18).

**Remark 4.2.** A rational analog of Theorem A is not possible. That is, there exist  $B\Gamma \xrightarrow{f} BK_{\mathbb{Q}}^{\wedge}$  that are not null while  $f|_{BT}$  is null. If  $\Gamma$  is compact connected Lie, then  $f|_{BT}$  is null implies that  $f$  is null by obstruction theory. However, the uniqueness obstructions to extending  $0 \simeq f|_{B\Gamma_I}$  for  $I \in \mathbf{S}$  to  $B\Gamma$  do not vanish in general. See (24) and Section 7 below. Moreover,  $\text{map}(B\Gamma, BK_{\mathbb{Q}}^{\wedge})_0$  is not always simply-connected by (23).  $\square$

## 5. APPLYING THE ARITHMETIC FIBER SQUARE

We now begin our applications of Theorem A to integral maps  $B\Gamma \xrightarrow{f} BK$  where  $\Gamma$  is a connected compact Lie or Kac–Moody group with maximal torus  $\iota: T \rightarrow \Gamma$  and  $K$  is a Kac–Moody group. Because  $BK$  is a simply-connected  $CW$ -complex [33], it is given as a homotopy pullback

$$(20) \quad \begin{array}{ccc} BK & \xrightarrow{\Pi(-)_p^{\wedge}} & \prod BK_p^{\wedge} \\ \downarrow (-)_{\mathbb{Q}}^{\wedge} & & \downarrow (-)_{\mathbb{Q}}^{\wedge} \\ BK_{\mathbb{Q}}^{\wedge} & \xrightarrow{[\Pi(-)_p^{\wedge}]_{\mathbb{Q}}^{\wedge}} & \prod (BK_p^{\wedge})_{\mathbb{Q}}^{\wedge} \end{array}$$

known as the arithmetic fiber square [15] where  $(-)_{\mathbb{Q}}^{\wedge}$  denotes rationalization. By applying  $\text{map}(B\Gamma, -)$  to the above diagram, we have a homotopy pullback of mapping spaces. In this section, we obtain explicit obstructions to extending a null homotopic map  $BT \xrightarrow{0} BK$  to  $B\Gamma \xrightarrow{f} BK$  uniquely. In other words, we give a description of the set homotopy classes of  $B\Gamma \xrightarrow{f} BK$  whose restriction to  $BT$  is null. Broadening speaking, we now follow the proofs of [27, Theorems 3.1 and 3.11(ii)].

It turns out that—as in [2, 27]— $f|_{BT}$  is null implies that  $f_{\mathbb{Q}}^{\wedge}$  is null.

**Lemma 5.1** (cf. [27, Proof of Theorem 3.1]). *The canonical map*

$$(21) \quad [B\Gamma, BK_{\mathbb{Q}}^{\wedge}] \rightarrow [B\Gamma, \prod (BK_p^{\wedge})_{\mathbb{Q}}^{\wedge}]$$

*is an injection.*

Though we are most concerned with the null fiber of (21), all fibers are of general interest. In particular, if  $f_p^{\wedge}$  is null for all primes  $p$ , then

$$B\Gamma \xrightarrow{f} BK \rightarrow \prod (BK_p^{\wedge})_{\mathbb{Q}}^{\wedge}$$

is null so that Lemma 5.1 implies  $f_{\mathbb{Q}}^{\wedge}$  is nullhomotopic. Recall that Lemma 5.1 holds whenever  $BK_{\mathbb{Q}}^{\wedge}$  is a product of  $K(\mathbb{Q}, n)$ ’s—as in [2, 27]. In Section 7 below, we see that  $BK_{\mathbb{Q}}^{\wedge}$  is very often not such a product. A proof of Lemma 5.1 given in Section 6.

Considering Theorem A and the preceding discussion, we will now restrict our attention to maps  $B\Gamma \xrightarrow{f} BK$  whose  $p$ -completions  $f_p^{\wedge}$  and rationalization  $f_{\mathbb{Q}}^{\wedge}$  are all nullhomotopic. Define  $N_K^{\Gamma} \subseteq [B\Gamma, BK]$  to be the corresponding set of homotopy classes. We can now summarize our discussion in the following technical lemma.

**Lemma 5.2.** *The set  $N_K^{\Gamma}$  is in natural bijection with*

$$(22) \quad V_K^{\Gamma} := \pi_1(\text{map}(B\Gamma, BK_{\mathbb{Q}}^{\wedge}), 0) \setminus \pi_1(\text{map}(B\Gamma, \prod (BK_p^{\wedge})_{\mathbb{Q}}^{\wedge}), 0)$$

*so that the class of nullhomotopic maps corresponds to  $[1] \in V_K^{\Gamma}$  and any map  $B\Gamma' \xrightarrow{g} B\Gamma$  induces  $N_K^{\Gamma} \xrightarrow{g_{\Gamma'}^{\Gamma}} N_K^{\Gamma'}$ .*

**Proof:** The homotopy Bousfield–Kan spectral sequence [4, XI. §7] implies that  $N_K^\Gamma$  is naturally isomorphic to  $\lim^1 \pi_1(\text{map}(B\Gamma, P)_0)$  where  $P$  is the homotopy pullback diagram (20) for  $BK$ . In particular,  $N_K^\Gamma$  is naturally isomorphic to  $V_K^\Gamma$  since  $\pi_1(\text{map}(B\Gamma, \coprod BK_p^\wedge), 0)$  vanishes by Theorem A.  $\square$

In the more restricted contexts of [2, 27], all  $V_K^\Gamma$  are trivial. Here we identify  $V_K^\Gamma$  as an (often nontrivial) abelian group so that any  $g_\Gamma^\Gamma$  corresponds to a group homomorphism. Hence, the kernel of  $V_K^T \rightarrow V_K^\Gamma$  induced by the standard torus corresponds to elements of  $N_K^\Gamma$  to restrict to zero in  $[BT, BK]$ .

First note

$$\begin{aligned} \pi_1(\text{map}(B\Gamma, BK_\mathbb{Q}^\wedge), 0) &\cong [B\Gamma, \Omega(BK_\mathbb{Q}^\wedge)] \\ &\cong [B\Gamma, K_\mathbb{Q}^\wedge] \\ (23) \quad &\cong \prod_{j \in J} H^{n_j}(B\Gamma, \mathbb{Q}) \end{aligned}$$

where the last identification follows from the fact that  $K$  is an  $H$ -space and a finite type  $CW$ -complex (see, e.g., [31, I: §2-1] and [33, 12.2]). That is,  $H^*(K, \mathbb{Q})$  is free and  $K_\mathbb{Q}^\wedge$  is a countable product of Eilenberg–MacLane spaces (see also Proposition A.2 below). Setting  $\hat{\mathbb{Q}} := (\prod \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$  we have

$$\pi_1(\text{map}(B\Gamma, \prod (BK_p^\wedge)_\mathbb{Q}^\wedge), 0) \cong \prod_{j \in J} H^{n_j}(B\Gamma, \hat{\mathbb{Q}})$$

as in (23). In conclusion,

$$(24) \quad V_K^\Gamma \cong \prod_{j \in J} H^{n_j}(B\Gamma, \hat{\mathbb{Q}}/\mathbb{Q})$$

and we can identify the kernel of  $V_K^T \rightarrow V_K^\Gamma$ .

**Theorem 5.3.** *If  $BT \xrightarrow{B\iota} B\Gamma \xrightarrow{f} BK$  is nullhomotopic, then the group of obstructions to  $f$  being null is isomorphic to*

$$(25) \quad \prod_{j \in J_{\text{even}}} \ker(H^{n_j}(B\iota^*, \hat{\mathbb{Q}}/\mathbb{Q})) \times \prod_{j \in J_{\text{odd}}} H^{n_j}(B\Gamma, \hat{\mathbb{Q}}/\mathbb{Q})$$

where  $J_{\{\text{odd}, \text{even}\}}$  is the set of  $\{\text{odd}, \text{even}\}$   $n_j$  (with multiplicity) from (23) and  $|J_{\text{odd}}|$  is at most the rank of  $K$ . In particular, the set of homotopy classes of such  $f$  is isomorphic to the above abelian group.

**Proof:** Noting that  $H^*(BT, \hat{\mathbb{Q}}/\mathbb{Q})$  is concentrated in even degrees, it remains only to show that  $|J_{\text{odd}}|$  is at most the rank of  $K$  by the preceding discussion. Recall the split short exact sequence of Hopf algebras [33]

$$(26) \quad 1 \rightarrow H^*(K/T_K, \mathbb{Q}) \otimes_S \mathbb{Q} \rightarrow H^*(K, \mathbb{Q}) \rightarrow \Lambda(x_{n_1}, \dots, x_{n_k}) \rightarrow 1$$

where  $T_K$  denotes the standard torus of  $K$ , the right algebra is exterior on odd degree generators,  $k$  is bounded by the rank of  $T_K$ , and the left algebra is concentrated in even degrees (and hence polynomial).  $\square$

Theorem B follows easily from Theorem 5.3.

**Proof of Theorem B:** If  $\Gamma = G$  is a connected compact Lie group with Weyl group  $W$  and rank  $r$ , then  $H^*(BG, \hat{\mathbb{Q}}/\mathbb{Q}) = \hat{\mathbb{Q}}/\mathbb{Q}[t_1, \dots, t_r]^W$  is concentrated in even degrees and  $H^*(B\iota^*, \hat{\mathbb{Q}}/\mathbb{Q})$  corresponds to the inclusion of  $W$ -invariants. Thus, the uniqueness obstructions (25) vanish as desired.  $\square$

As detailed in Section 6 below,  $\ker(H^*(B\iota^*, \hat{\mathbb{Q}}/\mathbb{Q}))$  is the subring of nilpotent elements of  $H^*(B\Gamma, \hat{\mathbb{Q}}/\mathbb{Q})$ . In this way, Theorem 5.3 describes how obstructions result from the existence nilpotent elements in certain degrees depending on  $K$ .

**Example 5.4.** To see that non-trivial obstructions can occur, let  $K(A)$  be the affine Kac–Moody group associated to the generalized Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Here it is known that  $H^*(K(A), \mathbb{Q}) = \Lambda(c_1, y_3, y_5) \otimes \mathbb{Q}[x_4]$  as an algebra where subscripts denote degree. Using the methods developed in the next two sections, we can calculate

$$\begin{aligned} V_{K(A)}^{K(A)} &\cong \prod_{i \in \{1, 3, 4, 5\}} H^i(BK(A), \hat{\mathbb{Q}}/\mathbb{Q}) \\ &\cong H^4(BK(A), \hat{\mathbb{Q}}/\mathbb{Q}) \times H^5(BK(A), \hat{\mathbb{Q}}/\mathbb{Q}) \cong (\hat{\mathbb{Q}}/\mathbb{Q})^3; \end{aligned}$$

see Example 7.6. This implies

$$\ker \left( V_{K(A)}^T \rightarrow V_{K(A)}^{K(A)} \right) \cong H^5(BK(A), \hat{\mathbb{Q}}/\mathbb{Q}) \cong \hat{\mathbb{Q}}/\mathbb{Q}.$$

Hence, each non-zero  $x \in \hat{\mathbb{Q}}/\mathbb{Q}$  is represented by a non-trivial  $f_x: BK(A) \rightarrow BK(A)$  such that  $f_x B\iota$  is null.

We can also construct a non-empty set of  $f_x: BK(A) \rightarrow BG$  with these properties such that  $G$  is compact connected Lie. That is,  $\ker \left( V_G^T \rightarrow V_G^{K(A)} \right)$  is nontrivial whenever  $H^*(BG, \mathbb{Q})$  has a polynomial generator in degree 6. Consider, for example,  $G = SU(n)$  for  $n \geq 3$ .  $\square$

## 6. VANISHING RESULTS AND THE PROOF OF LEMMA 5.1

This section gives vanishing results for the left or right terms of (25) in Theorem 5.3 and a proof Lemma 5.1. To these ends, we will maintain the notations of the previous section including letting  $B\Gamma \xrightarrow{f} BK$  denote a map between classifying spaces with  $\Gamma$  is a connected compact Lie or Kac–Moody group and  $K$  a Kac–Moody group. Placing conditions on the source  $B\Gamma$  will guarantee that the left term of (25) vanishes while placing conditions on the target  $BK$  will guarantee that right term vanishes so that—under these conditions— $f$  is nullhomotopic if and only if  $f|_{BT}$  is null.

Our line of attack uses the homotopy colimit presentation for  $B\Gamma$ . Specifically, our discussion here centers around computing  $[B\Gamma, BK_{\mathbb{Q}}^{\wedge}]$  in terms of  $\text{map}(B\Gamma_I, BK_{\mathbb{Q}}^{\wedge})$  and  $H^*(B\Gamma, \mathbb{Q})$  in terms of  $H^*(B\Gamma_I, \mathbb{Q})$  via well-known spectral sequences. We note that the rational cohomology of the classifying space of a Kac–Moody group is not well-studied and refer the reader to [33] and Section 7 for examples.

Let us fix a  $BT \xrightarrow{f} BK_{\mathbb{Q}}^{\wedge}$  and consider the problem of extending  $f$  to  $B\Gamma$ . For computation, we appeal to the homotopy Bousfield–Kan spectral sequence [4, XI. §7] associated to the functor  $\text{map}(B\Gamma_I, Z)$  has second page given by

$$(27) \quad E_2^{i,k} = \lim_{\mathbf{S}^{\text{op}}}^i \pi_k(\text{map}(D_{\Gamma}, Z)_{\overline{f}})$$

where we take  $Z = BK_{\mathbb{Q}}^{\wedge}, (\prod BK_p^{\wedge})_{\mathbb{Q}}^{\wedge}$  and  $\overline{f}$  denotes the set of homotopy classes that extend  $f$ . Its related cohomological spectral sequence [4, XII. 5.8] for computing  $H^*(B\Gamma, L)$  has second page given by

$$(28) \quad F_2^{i,k} = \lim_{\mathbf{S}^{\text{op}}}^i H^k(B\Gamma, L)$$

where we take  $L = \mathbb{Q}, \hat{\mathbb{Q}}$ . We will use the notation  $E_n^{i,k}$  and  $F_n^{i,k}$ , respectively, for these spectral sequences. The proof of Lemma 5.1 depends on comparing them.

In fact,  $F_n^{i,k}$  has already been studied in some detail in [34].

**Theorem 6.1** (cf. [34, 2.3, 2.5]). *For any Kac–Moody group  $\Gamma$  with Weyl group  $W$  and  $L = \mathbb{Q}, \hat{\mathbb{Q}}, F_n^{*,*}$  collapses at  $F_2^{*,*}$  and there are canonical isomorphisms*

$$(29) \quad F_2^{i,k} \cong H^i((W, H^k(BT, L))).$$

**Proof:** For the collapse of  $F_n^{*,*}$  at  $F_2^{*,*}$ , see the proof of [34, 2.3]. See the proof of [34, 2.5] for the identification (29).  $\square$

Moreover, by making suitable identifications, the collapse of  $F_\infty^{*,*}$  implies the collapse of  $E_\infty^{*,*}$ .

**Theorem 6.2.** *With the preceding notations, we have  $E_2^{*,*} = E_\infty^{*,*}$  for  $Z = BK_{\mathbb{Q}}^\wedge, (\prod BK_p^\wedge)_{\mathbb{Q}}^\wedge$ .*

Before giving the proof of Theorem 6.2, let us first provide further motivation by relating the left hand term of (25) to  $F_2^{i,k}$ .

**Proposition 6.3.** *With the preceding notation, the following are equivalent:*

- (i) *left hand term of (25) vanishes*
- (ii)  *$H^{n_j}(B\Gamma, \mathbb{Q})$  has no nilpotent elements for all even  $n_j$  from (23)*
- (iii)  *$F_2^{2i, n_j-2i} = \lim^{2i} H^{n_j-2i}(D_\Gamma, \mathbb{Q}) = 0$  for all even  $n_j$  and  $i \geq 1$*

**Proof:** First note that  $H^*(\iota, \mathbb{Q})$  factors through the edge homomorphism

$$(30) \quad F_\infty^{*,*} \rightarrow F_\infty^{0,*} \cong H^0(W, H^*(BT, \mathbb{Q})) \cong \mathbb{Q}[t_1, \dots, t_r]^W$$

where  $\iota$  is the maximal torus of rank  $r$ . Because  $F_2^{i,k}$  carries a bigraded multiplicative structure, (ii) and (iii) are equivalent noting Theorem 6.1. Specifically, elements in the zero column are not nilpotent and elements in positive columns have nilpotent degree bounded by the maximal chain length in  $\mathbf{S}$ . Likewise, (i) and (iii) are equivalent because kernel (30) is precisely the nonzero columns.  $\square$

Since we are currently focused on vanishing, we recall the following definition.

**Definition 6.4.** A Kac–Moody group  $K$  is  $n$ -spherical if in its canonical homotopy colimit presentation (2) the elements of the poset of spherical subsets  $\mathbf{S}$  have cardinality at most  $n$ .

If  $K$  is  $n$ -spherical, then chains in  $\mathbf{S}$  have length at most  $n$  so that  $\lim_{\mathbf{S}}^{i>n} M = 0$  for *any* diagram  $M : \mathbf{S}^{op} \rightarrow \mathbf{Abelian}$  of abelian groups. For instance, if  $K$  is 3-spherical, then the last condition in Proposition 6.3 is simply to  $\lim^2 H^{n_j-2}(D_\Gamma, \mathbb{Q}) = 0$  for all even  $n_j$  from (23). Of course, this vanishing is immediate if  $K$  is 1-spherical. More explicitly, if  $\lim_{\mathbf{S}^{op}}^{i \geq 2} H^*(\Gamma, \mathbb{Q})$ , then  $F_2$  yields a short exact sequence

$$(31) \quad \mathbb{Q}[t_1, \dots, t_r]^W = \lim_{\mathbf{S}^{op}}^0 H^*(D_\Gamma, \mathbb{Q}) \rightarrow H^*(B\Gamma, \mathbb{Q}) \rightarrow \lim_{\mathbf{S}}^1 H^*(D_\Gamma, \mathbb{Q}).$$

As we have seen, this implies that the left hand term of (25) vanishes.

If we can replace  $D_\Gamma$  with a subdiagram over a subposet with a smaller chain length bound, then this subdiagram provides a more favorable  $E_2$ . For this purpose, we make the following definition.

**Definition 6.5.** A Kac–Moody group  $\Gamma$  is essentially  $n$ -spherical if there exists a right cofinal full subposet  $\mathbf{S}' \subseteq \mathbf{S}$  whose chains have length at most  $n$ .

In lieu of a definition of right cofinal (see, e.g. [26, 19.6.1]), we give the two properties relevant to our discussion and a practical criterion for checking if  $\Gamma$  is essentially  $n$ -spherical. Specifically, a right cofinal  $\mathbf{S}' \subseteq \mathbf{S}$  induces a canonical homotopy equivalence (see, e.g. [26, 19.6.17])

$$(32) \quad \text{hocolim}_{I \in \mathbf{S}'} B\Gamma_I \xrightarrow{\sim} \text{hocolim}_{I \in \mathbf{S}} B\Gamma_I \xrightarrow{\sim} BK$$

so that Theorem 6.1 implies that the canonical comparison maps

$$(33) \quad \lim_{(\mathbf{S}')^{\text{op}}}^i H^*(D_\Gamma, \mathbb{Q}) \xrightarrow{\sim} \lim_{\mathbf{S}^{\text{op}}}^i H^*(D_\Gamma, \mathbb{Q})$$

are isomorphisms for all  $i \geq 0$  (see also, e.g., [45, 1.5]). The particularities of our situation yield the following.

**Proposition 6.6.** *For a Kac–Moody group  $K$  with spherical subsets  $\mathbf{S}$ , there exists a unique minimal, right cofinal, full subposet  $\overline{\mathbf{S}} \subseteq \mathbf{S}$  whose elements are intersections of maximal elements of  $\mathbf{S}$ . In particular,  $K$  is essentially  $n$ -spherical if and only if all sets of  $n$  distinct maximal elements of the poset of spherical subsets have a common intersection.*

**Proof:** For any finite poset, the existence of a unique minimal, right cofinal, full subposet is guaranteed by [22, Theorem 4.3]. For  $\mathbf{S}$ , it is convenient to identify this subposet by considering covers of categories [17, 2.5].

To see that  $\overline{\mathbf{S}} \subseteq \mathbf{S}$  is right cofinal, note that the maximal elements provide a canonical cover of  $\mathbf{S}$  and  $\mathbf{S}$  has greatest lower bounds. That is, the collection of  $\mathbf{S} \downarrow I_m \subseteq \mathbf{S}$  defined as the full subposet of elements under a fixed maximal  $I_m \in \mathbf{S}$  is a cover and  $\overline{\mathbf{S}} \subseteq \mathbf{S}$  factors as a composition of cofinal functors  $\overline{\mathbf{S}} \rightarrow \overline{\mathbf{S}} \times \mathbf{S} \downarrow I \rightarrow \mathbf{S}$  [17, Proposition 2.5.3].

To see that  $\overline{\mathbf{S}}$  is minimal, we must show that  $I \notin \overline{\mathbf{S}}$  implies that the full subposet of elements strictly greater than  $I$  denoted  $I \downarrow \downarrow \mathbf{S}$  has a contractible realization [22, Theorem 4.3]. Since  $I \notin \overline{\mathbf{S}}$ , the maximal elements of  $I \downarrow \downarrow \mathbf{S}$  constitute a non-empty subset of the maximal elements of  $\mathbf{S}$  and provide a cover of  $I \downarrow \downarrow \mathbf{S}$  as above. The analogous composition of cofinal functors now shows that the realization of  $I \downarrow \downarrow \mathbf{S}$  is homotopic to a  $(k-1)$ -simplex where  $k$  is the number of maximal elements in  $I \downarrow \downarrow \mathbf{S}$ .

Clearly, the maximal chain length  $l$  of  $\overline{\mathbf{S}}$  is the minimal  $n$  such that  $K$  is essentially  $n$ -spherical. The initial object of  $\overline{\mathbf{S}}$  is necessarily the intersection of all maximal elements of  $\mathbf{S}$  and must coincide with the intersection of all sets of  $l$  distinct all maximal elements of  $\mathbf{S}$ .  $\square$

Let us now consider the right hand term of (25). We already have seen that  $J_{\text{odd}}$  is finite and when  $K$  is indefinite indecomposable, we have further control on  $J_{\text{odd}}$ .

**Proposition 6.7.** *If  $K$  is indefinite indecomposable, then the right hand term of (25) vanishes—without further restrictions on  $\Gamma$ .*

**Proof:** For indefinite, indecomposable  $K$ , odd  $n_j$  equal one or three by [30, Appendix], [48] (see also Appendix A). Since  $B\Gamma$  is simply-connected, it remains to show that  $H^3(B\Gamma, \mathbb{Q}) = 0$ . By applying the the Rothenberg–Steenrod spectral sequence to  $H^*(\Gamma, \mathbb{Q}) = 0$  (see, e.g., [31, II: §6-2 Theorem B (ii)]), this vanishing is easily verified by recalling (26) and checking that  $H^2(\Gamma/T, \mathbb{Q}) \otimes_S \mathbb{Q} = 0$ . This latter vanishing follows, for example, from the fact that degree two elements in  $S^2$  generate  $H^2(\Gamma/T, \mathbb{Q})$  [33, §6].  $\square$

We can now summarize our discussion with the following theorem.

**Theorem 6.8.** *If  $K$  is indefinite, indecomposable, then the following are equivalent:*

- (i)  $B\Gamma \xrightarrow{f} BK$  is null if and only if  $f|_{BT}$  is null
- (ii)  $H^{n_j}(B\Gamma, \mathbb{Q})$  has no nilpotent elements for all even  $n_j$  from (23)
- (iii)  $F_2^{2i, n_j-2i} = \lim^{2i} H^{n_j-2i}(D_\Gamma, \mathbb{Q}) = 0$  for all even  $n_j$  and  $i \geq 1$

*In particular, these conditions are equivalent to  $\lim^2 H^{n_j-2}(D_\Gamma, \mathbb{Q}) = 0$  when  $\Gamma$  is essentially 3-spherical and always hold when  $\Gamma$  is essentially 1-spherical.*

The proof of Theorem C has now been reduced to a nilpotency condition on rational cohomology. Over the next two sections, we will prove the following.

**Theorem 6.9.** *If a Kac-Moody group  $K$  is 2-spherical, then  $\lim^2 H^*(D_K, \mathbb{Q}) = 0$  where  $D_K$  is the diagram for its canonical homotopy colimit presentation (2).*

Hence, combining the above with Theorem 6.8 proves Theorem C. More generally, the following is immediate.

**Theorem 6.10.** *Under either of the following two additional hypotheses:*

- $\Gamma$  is connected compact Lie
- $\Gamma$  is a 2-spherical Kac-Moody group and  $K$  has no affine or Lie direct product factors,

*$f: B\Gamma \rightarrow BK$  is nullhomotopic  $\iff f|_{BT}$  is nullhomotopic.*

To see that this hypothesis on  $K$  is necessary, note Example 7.1 below.

We now turn to the proof of Lemma 5.1 which is much easier if  $\Gamma$  is essentially 1-spherical and motivates our approach.

**Proposition 6.11.** *If  $\Gamma$  is essentially 1-spherical, then (21) is an injection.*

**Proof:** As in (31), we have the collapse at  $E_2$  of the associated homotopy spectral sequence (27) and we obtain a map of short exact sequences

$$\begin{array}{ccccc} \lim_{\mathbb{S}^{\text{op}}}^1 \pi_1(\text{map}(D_\Gamma, BK_{\mathbb{Q}}^\wedge)) & \longrightarrow & [B\Gamma, BK_{\mathbb{Q}}^\wedge] & \longrightarrow & \lim_{\mathbb{S}^{\text{op}}} [D_\Gamma, BK_{\mathbb{Q}}^\wedge] \\ \downarrow & & \downarrow & & \downarrow \\ \lim_{\mathbb{S}^{\text{op}}}^1 \pi_1(\text{map}(D_\Gamma, \prod (BK_p^\wedge)_{\mathbb{Q}}^\wedge)) & \longrightarrow & [B\Gamma, \prod (BK_p^\wedge)_{\mathbb{Q}}^\wedge] & \longrightarrow & \lim_{\mathbb{S}^{\text{op}}} [D_\Gamma, \prod (BK_p^\wedge)_{\mathbb{Q}}^\wedge]. \end{array}$$

In particular, it is sufficient to show that the leftmost vertical map is an injection because the rightmost vertical map corresponds to an inclusion of  $W$ -fixed points

$$\mathbb{Q}[t_1, \dots, t_r]^W \hookrightarrow \hat{\mathbb{Q}}[t_1, \dots, t_r]^W.$$

By (23) and (31), this map of  $\lim^1$  terms corresponds to a change in coefficients

$$\prod_{j \in J_{\text{even}}} H^{n_j+1}(B\Gamma, \mathbb{Q} \hookrightarrow \hat{\mathbb{Q}}).$$

□

Using Theorem 6.1, this argument can be refined to handle the general case.

**Proof of Theorem 6.2:** Consider the  $E_2^{*,0}$ . The uniqueness obstructions to extending  $f$  to  $B\Gamma_I$  are canonically isomorphic to  $H^i(W_I; \pi_i(\text{map}(BT, Z)_f))$  and vanish. Hence  $E_2^{i,0} = 0$  for  $i \geq 1$ .

Now let  $F_n^{*,*}$  be the associated spectral cohomology sequence (28) with coefficients in  $L = \mathbb{Q}, \hat{\mathbb{Q}}$ . For  $n \geq 2$ , the differentials  $e_n$  on  $F_n^{*,*}$  closely related to the differentials  $d_n$  on  $E_n^{*,*}$  by (23). That is,  $E_2^{i,k}$  is given by

$$\lim^i \pi_k(\text{map}(D_\Gamma, Z)) \cong \lim^i \prod_{j \in J_{\text{odd}}} H^{n_j-k+1}(D_\Gamma, L) \cong \prod_{j \in J_{\text{odd}}} \lim^i H^{n_j-k+1}(D_\Gamma, L)$$

for  $k \geq 1$  whereas  $F_2^{i,k}$  is given by  $\lim^i H^j(D_\Gamma, L)$ . By definition [4, XII. 5.8],  $d_2^{i,k}$  is given as a product of  $e_2^{i, n_j-k+1}$  wherever the source or target is not in the zero column. Equivalently, we can consider (the shift of) the spectrum associated to  $K_L^\wedge$ . In particular,  $d_n^{i,k} = 0$  for all  $n \geq 2$  by the collapse of  $F_n^{*,*}$  ( $k \geq 1$ ) and the first paragraph of this proof ( $k = 0$ ). □

**Proof of Lemma 5.1:** The map (21) is induced by a map on  $E_2 = E_\infty$  (27) associated to  $BK_{\mathbb{Q}}^\wedge \rightarrow (\prod BK_p^\wedge)_{\mathbb{Q}}^\wedge$ . Moreover, (21) is an injection if and only if the restriction of this map of  $E_\infty$ -pages to total degree zero is an injection.

The change in coefficients  $\mathbb{Q} \hookrightarrow \hat{\mathbb{Q}}$  induces an injection on cohomology. Therefore, the map induced on  $F_2 = F_\infty$  is an injection. By the above identifications, the

corresponding map of the total degree zero parts of  $E_\infty$ -pages must be an injection. Thus, (21) is an injection as desired.  $\square$

## 7. EXPLICIT $\lim^2$ COMPUTATIONS AND EXAMPLES

If a Kac–Moody group  $K$  is 2-spherical, then we saw in the previous section that—since maximal chain length in  $\mathbf{S}$  is at most two—Theorem C is equivalent to the vanishing of  $\lim^2 H^{n_j-2}(D_K, \mathbb{Q})$  for all even  $n_j$  from (23) (see Theorem 6.8). In this section, we give a method to compute  $\lim_{\mathbf{S}}^2 M$  for an arbitrary diagram of free  $R$ -modules  $M: \mathbf{S}^{op} \rightarrow R\text{-mod}$  over some fixed ring  $R$ .

This method is applied in the next section to prove Theorem 6.9 and obtain the desired vanishing. For 2-spherical  $K$ , Theorem 6.9 also implies that the short exact sequence (31) is available to compute  $H^*(BK, \mathbb{Q})$ . The second purpose of this section is to give examples where (31) computes  $H^*(BK, \mathbb{Q})$  as a ring. Let us now give a 1-spherical example.

**Example 7.1.** Taking, for example, the non-symmetrizable, non-singular matrix

$$A_1 = \begin{bmatrix} 2 & -1 & -1 \\ -7 & 2 & -1 \\ -8 & -9 & 2 \end{bmatrix},$$

$K := K(A_1)$  is indefinite, indecomposable, and 1-spherical with rank 3 torus. These conditions imply that  $\mathbb{Q}[t_1, t_2, t_3]^W = \mathbb{Q}$  (see Table 1) and  $H^*(BK, \mathbb{Q})$  is necessarily concentrated in odd degrees by (31). Moreover, a simple Euler characteristic calculation gives the dimension of each odd degree. Specifically, it is straightforward to compute the Poincaré series of each degree of the standard normalized chain complex whose homology computes  $\lim_{\mathbf{S}}^i H^*(BK_I, \mathbb{Q})$  (see, e.g., [21, Appendix II.3] and [47, 8.3] or [11, 2.3]). Because  $\lim_{\mathbf{S}}^0 H^{*\geq 1}(BK_I, \mathbb{Q}) = 0$ , the degree zero and one Poincaré series give the Poincaré series for  $\lim_{\mathbf{S}}^1 H^*(BK_I, \mathbb{Q})$ . In this way, the Poincaré series for  $H^{*\geq 1}(BK, \mathbb{Q})$  is given by:

$$\begin{aligned} t + \frac{2t}{(1-t^2)^3} - \frac{3t}{(1-t^2)^2(1-t^4)} &= t + \sum_{n=1}^{\infty} \frac{2n^2 - 5 - 3(-1)^n}{8} t^{2n+1} \\ &= 2t^7 + 3t^9 + 6t^{11} + 8t^{13} + 12t^{15} + 15t^{17} + \dots \end{aligned}$$

In particular,  $H^*(BK, \mathbb{Q})$  is infinitely generated and  $BK_{\mathbb{Q}}^{\wedge}$  is not a product of Eilenberg–MacLane spaces. Indeed, the product of any two odd degree elements has positive, even degree and necessarily vanishes. This determines  $H^*(BK, \mathbb{Q})$  as a ring. The careful reader will have already noted that it is the properties stated above rather than the specific form of  $A_1$  that are needed for this computation of degrees.

We emphasize that with  $H^*(BK, \mathbb{Q})$  in hand we can read off the set of homotopy classes of  $f_x: BK \rightarrow BG$  such that  $G$  is compact connected Lie and  $f_x B\iota$  is null from (25). For instance, if  $G = SU(n+1)$ ,  $G_{\mathbb{Q}}^{\wedge} \simeq K(\mathbb{Q}, 3) \times \dots \times K(\mathbb{Q}, 2n+1)$ , then group of uniqueness obstructions given by (25) is

$$H^3(BK, \hat{\mathbb{Q}}/\mathbb{Q}) \times \dots \times H^{2n+1}(BK, \hat{\mathbb{Q}}/\mathbb{Q}) \cong (\hat{\mathbb{Q}}/\mathbb{Q})^{\lfloor \frac{2n^3+3n^2-4n}{24} \rfloor + (-1)^{n+1}}$$

That is, up to isomorphism, the sequence of obstruction groups begins

$$\left\{ \ker \left( V_{SU(n+1)}^T \rightarrow V_{SU(n+1)}^K \right) \right\}_{n \geq 1} = 0, 0, (\hat{\mathbb{Q}}/\mathbb{Q})^2, (\hat{\mathbb{Q}}/\mathbb{Q})^5, (\hat{\mathbb{Q}}/\mathbb{Q})^{11}, (\hat{\mathbb{Q}}/\mathbb{Q})^{19}, \dots$$

Similarly, it is straightforward to exhibit affine Kac–Moody groups  $K'$  such that homotopically nontrivial  $f_x: BK(A_1) \rightarrow BK'$  with  $f_x B\iota$  nullhomotopic exist. Consider, for example, the (untwisted) affine Kac–Moody group  $K'$  obtained by extending  $SU(n+1)$  for  $n \geq 3$  [35, XIII].  $\square$



As in Example 7.1, our computation a  $\lim_{\mathbf{S}}^2 M$  for some diagram of free  $R$ -modules  $M: \mathbf{S}^{op} \rightarrow R\text{-mod}$  begins with the standard normalized chain complex associated to  $M$ . First, we give a concrete description of  $\lim_{\mathbf{S}}^2 M$  in a special case. Second, we show that  $\lim_{\mathbf{S}}^2 M$  surjects onto  $\prod \lim_{\mathbf{S}_k}^2 M$  for a finite collection subposets  $\mathbf{S}_k \subseteq \mathbf{S}$  such that terms of this product can be described as in the special case and the kernel of this surjection is explicit.

Our arguments employ algebraic discrete Morse theory and depend only on the form of  $\mathbf{S}$ . In particular, they are relevant to essentially 2-spherical Kac-Moody groups as well as situations unrelated to Kac-Moody groups where  $\lim^2 M$  is nonzero. For further details on (algebraic) discrete Morse theory, we give specific references in [11, 19, 28, 44]. We are pleased to recommend [11, Sections 2-3] in particular for a reasonably self-contained introduction that is formulated cohomologically. Our applications of discrete Morse theory will be fairly elementary and motivating figures will be given.

The following lemma constitutes the first step.

**Lemma 7.2.** *Define the poset of subsets of  $\{1, \dots, n\}$*

$$(34) \quad C_n = \{\emptyset, \{1\}, \dots, \{n\}, \{1, 2\}, \dots, \{n-1, n\}, \{n, 1\}\}$$

*ordered by inclusion. For any  $n \geq 3$  and diagram of free  $R$ -modules,  $M: C_n^{op} \rightarrow R\text{-mod}$*

$$(35) \quad \lim_{C_n}^2 M \cong M_{\emptyset} / \langle M_1^{\emptyset}, \dots, M_n^{\emptyset} \rangle$$

*where  $M_I := M(I)$ ,  $M_i := M(\{i\})$ , and  $M_i^{\emptyset} := \text{Im}(M(\{i\} \rightarrow \emptyset))$ .*

**Proof:** Here the standard normalized chain complex has the form

$$0 \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow 0$$

with  $C^n = \prod_{I_0 \subsetneq \dots \subsetneq I_n} M(I_0)$  and is chain homotopic to

$$0 \rightarrow C^0 / M_{\emptyset} \rightarrow \prod_{1 \leq i \leq n} M_i \times M_i \rightarrow M_{\emptyset} \rightarrow 0$$

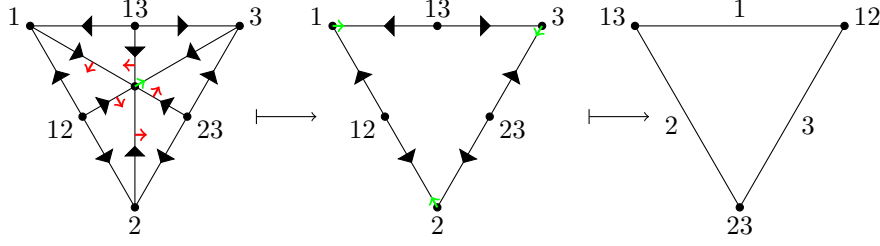
so that  $H^2(C^*)$  is the cokernel of  $M_1 \times M_1 \times \dots \times M_n \times M_n \rightarrow M_{\emptyset}$  given by

$$(36) \quad (x_1, \overline{x}_1, \dots, x_n, \overline{x}_n) \mapsto x_1^{\emptyset} + \overline{x}_1^{\emptyset} + \dots + x_n^{\emptyset} + \overline{x}_n^{\emptyset}$$

where  $x_i, \overline{x}_i \in M_i$  and  $x_i^{\emptyset}, \overline{x}_i^{\emptyset}$  are their respective images in  $M_{\emptyset}$  under  $M(\{i\} \rightarrow \emptyset)$ .

From the perspective of (algebraic) discrete Morse theory, this chain homotopy is induced by a single acyclic matching [28, 2.1], [11, 2.4] between  $M_{\emptyset}$  factors. Specifically, the unique  $M_{\emptyset}$  factor in  $C^0$  is paired with the  $M_{\emptyset}$  factor in  $C^1$  associated with the morphism  $\emptyset \rightarrow n$  and all other  $M_{\emptyset}$  factors in  $C^1$  factors are paired with  $M_{\emptyset}$  factors in  $C^2$  as follows: each  $\emptyset \rightarrow l$  coordinate for  $l < n$  pairs with the  $\emptyset \rightarrow l+1 \rightarrow \{l, l+1\}$  coordinate, each  $\emptyset \rightarrow \{l, l+1\}$  coordinate pairs with the  $\emptyset \rightarrow l \rightarrow \{l, l+1\}$  coordinate, and the  $\emptyset \rightarrow \{1, n\}$  coordinate pairs with the  $\emptyset \rightarrow 1 \rightarrow \{1, n\}$  coordinate. It is easy to check that this an acyclic pairing. For example, when  $n = 3$  the acyclic matching just described is depicted by the first step of Figure 1. Strictly speaking, the pairing we described involves choosing a basis for  $M_{\emptyset}$  and pairing these bases identically. The differential (36) is an instance of [28, 2.2], [11, 2.5].  $\square$

It is well-known that the chain complex can be further reduced. The second step in Figure 1 indicates an algebraic discrete Morse theory proof, but we do not spell this out. The smaller acyclic matching given is better motivation for later computations. Let us now give a 2-spherical example where Lemma 7.2 applies.

FIGURE 1. A picture of our discrete Morse theory pairing for  $\mathbf{C}_3$ .

**Example 7.3.** Taking, for example, the non-symmetrizable, non-singular matrix

$$A_2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -3 & 2 \end{bmatrix},$$

$K := K(A_2)$  is indefinite, indecomposable, and 2-spherical with rank 3 torus  $T := K_\emptyset$ . In particular,  $\mathbf{C}_3$  is the set of spherical subsets  $\mathbf{S}$  associated to  $K$  so that Lemma 7.2 gives the identification

$$\lim_{\mathbf{S}}^2 H^n(D_K, \mathbb{Q}) = H^n(BT, \mathbb{Q}) / \langle H^n(BK_{\{1\}}, \mathbb{Q}), \dots, H^n(BK_{\{3\}}, \mathbb{Q}) \rangle$$

In order to use the short exact sequence (31) as in Example 7.1, it remains to show that the  $\mathbb{Q}$ -linear span of  $\{H^n(BK_{\{i\}}, \mathbb{Q})\}_{i=1,2,3}$  is  $H^n(BT, \mathbb{Q})$ .

Since the Weyl group of  $K$  has an infinite set of reflecting hyperplanes, we can apply Lemma 8.6 with  $s$  generating the Weyl group of  $K_1$ ,  $t$  generating the Weyl group of  $K_2$  and  $u$  generating the Weyl group of  $K_3$ . (For reference, a discussion of the Weyl group of a Kac–Moody group is given in Section 8.) In particular,  $\{H^n(BK_{\{i\}}, \mathbb{Q})\}_{i=1,2,3}$  spans  $H^n(BT, \mathbb{Q})$  as desired.

So we have (31) as in Example 7.1. Hence,  $H^{\geq 1}(BK, \mathbb{Q})$  is necessarily concentrated in odd degrees, the cup product of any two odd degree elements is zero, and the rank of each odd degree is determined by an Euler characteristic calculation. In this way, the Poincaré series for  $H^*(BK, \mathbb{Q})$  is given by:

$$\begin{aligned} & 1 + t \left( 1 + \frac{-1}{(1-t^2)^3} + \frac{3}{(1-t^2)^2(1-t^4)} - \frac{1}{(1-t^2)(1-t^4)(1-t^6)} \right. \\ & \quad \left. - \frac{1}{(1-t^2)(1-t^4)(1-t^8)} - \frac{1}{(1-t^2)(1-t^4)(1-t^{12})} \right) \\ & = 1 + t^7 + t^9 + 3t^{11} + 2t^{13} + 5t^{15} + 7t^{17} + 7t^{19} + \dots \end{aligned}$$

and only one ring structure on  $H^*(BK, \mathbb{Q})$  is possible. Here the specific form of  $A_2$  effects the result, but the same method computes  $H^*(BK, \mathbb{Q})$  as a ring for any  $A_2$  with the properties stated above. As in Example 7.1,  $f: BK \rightarrow BK$  is null only if  $f|_{BT}$  is null, but it is straightforward to exhibit affine Kac–Moody  $K'$  or compact connected Lie  $G$  such that homotopically nontrivial  $g: BK \rightarrow BK'$  and  $h: BK \rightarrow BG$  with  $g|_{BT}$  and  $h|_{BT}$  nullhomotopic exist.  $\square$

Let us now consider the form of  $\mathbf{S}$  associated to a 2-spherical Kac–Moody group  $K$ . If the initial object  $\emptyset \in \mathbf{S}$  is removed, then the full subcategory obtained can be naturally associated with a simple graph  $(V_{\mathbf{S}}, E_{\mathbf{S}})$ . Let the vertices  $V_{\mathbf{S}}$  be given by  $\{1, \dots, r\}$  where  $r$  is the rank of the maximal torus and the edges  $E_{\mathbf{S}}$  be the cardinality two elements of  $\mathbf{S}$ . If, more generally,  $K$  is essentially 2-spherical Kac–Moody group  $K$ , then  $\overline{\mathbf{S}} \subseteq \mathbf{S}$  (defined in Proposition 6.6) is right cofinal with initial object  $\bullet$ . The full subcategory obtained by removing  $\bullet$  can be naturally associated

with a graph  $(V_{\bar{\mathbf{S}}}, E_{\bar{\mathbf{S}}})$ . The vertices  $V_{\bar{\mathbf{S}}}$  are minimal elements of  $\bar{\mathbf{S}} - \{\bullet\}$  while the edges  $E_{\bar{\mathbf{S}}}$  are maximal in  $\bar{\mathbf{S}} - \{\bullet\}$ . Roughly speaking, our strategy for describing  $\lim_{\bar{\mathbf{S}}}^2 M$  for some  $M : \bar{\mathbf{S}}^{op} \rightarrow R\text{-mod}$  will be to use a maximal spanning forest for  $(V_{\bar{\mathbf{S}}}, E_{\bar{\mathbf{S}}})$  in order to relate to cases covered by Lemma 7.2. Compare, for example, the construction of a fundamental basis for the first homology group of a simple graph [25, 4.6].

**Lemma 7.4.** *For any essentially 2-spherical Kac-Moody group with  $\bar{\mathbf{S}} \subseteq \mathbf{S}$ ,  $M : \bar{\mathbf{S}}^{op} \rightarrow R\text{-mod}$ , and (edge maximal) spanning forest of  $(V_{\bar{\mathbf{S}}}, E_{\bar{\mathbf{S}}})$ , there is a finite set of sub-posets  $\mathbf{S}_k \subseteq \bar{\mathbf{S}}$  which induces a canonical short exact sequence*

$$(37) \quad 0 \rightarrow \ker(\eta^*) \hookrightarrow \lim_{\bar{\mathbf{S}}}^2 M \xrightarrow{\eta^*} \prod_{\mathbf{S}_k} \lim_{\mathbf{S}_k}^2 M \rightarrow 0$$

where each  $\mathbf{S}_k \cong \mathbf{C}_{n_k}$  with  $n_k \geq 3$  and  $\ker(\eta^*)$  can be described explicitly (see (38)).

**Proof:** Let us abbreviate the notation  $(V_{\bar{\mathbf{S}}}, E_{\bar{\mathbf{S}}})$  to  $(V, E)$  in this proof. Define  $\dot{E} \subseteq E$  to be the edges not spanned by our hypothesized forest. For each  $e_k \in \dot{E} = \{e_1, \dots, e_m\}$ , there is a unique shortest path  $\gamma_k$  in the forest between the vertices of  $e_k$ . Define  $\mathbf{S}_k \subseteq \bar{\mathbf{S}}$  to be the full subposet with

$$\text{Objects}(\mathbf{S}) := \{\bullet, e_k\} \cup \{I \mid I \in \gamma_k\}$$

where  $I \in \gamma_k$  means that  $I \in \mathbf{S}$  is an edge or vertex in the path  $\gamma_k$ . Notice  $\mathbf{S}_k \cong \mathbf{C}_{n_k}$  for  $n_k$  one more than the length of  $\gamma_k$  and  $k \geq 3$  since  $(V, E)$  is simple. Now  $\eta^*$  is induced by the apparent map of posets

$$\prod_{k=1}^m \mathbf{S}_k \xrightarrow{\eta} \bar{\mathbf{S}}.$$

It remains to show that  $\eta^*$  is surjective and compute its kernel.

Any contraction of our hypothesized forest induces a homotopy from the metric graph realization of  $(V, E)$  to a disjoint union of bouquets of circles. In particular, we can choose a simplicial collapse of the simplicial realization  $(V, E)$  which can be described by (ordinary) discrete Morse theory [19]. See, e.g., Figure 2.

Also associated to this choice is a chain homotopy equivalence from the standard normalized chain complex

$$\begin{array}{ccccc} C^0 & \xrightarrow{\quad} & C^1 & \xrightarrow{\quad} & C^2 \\ \parallel & & \downarrow & & \downarrow \\ C^0 & \longrightarrow & \prod_{c \in \pi_0(V, E)} M_{t(c)} \times \prod_E M_v \times M_{\bullet} \times M_{\bar{v}} \times \prod_{\dot{E}} M_{\bullet} & \longrightarrow & \prod_{\dot{E}} M_{\bullet} \times M_{\bullet} \end{array}$$

where for an edge  $e \in E$  the pair  $\{v, \bar{v}\}$  is its vertices. This is depicted as the first stage of Figure 4. More specifically, this chain homotopy is induced by an acyclic pairing of  $M_{\bullet}$  coordinates from  $C^1$  and  $C^2$ . For each pairing of  $v$  and  $e = v \cup \bar{v}$  in the (ordinary) discrete Morse function on  $(V, E)$ , we make two pairings, namely: we pair the  $\bullet \subseteq v$  coordinate with the  $\bullet \subseteq v \subseteq e$  coordinate and we pair the  $\bullet \subseteq e$  coordinate with the  $\bullet \subseteq \bar{v} \subseteq e$  coordinate. It is immediate from the definitions that the acyclicity of the original Morse pairing implies that this (algebraic) pairing is acyclic.

As in Figure 4, we will also perform a second chain homotopy equivalence

$$\begin{array}{ccccccc} C^0 & \longrightarrow & \prod_{c \in \pi_0(V, E)} M_{t(c)} \times \prod_E M_v \times M_{\bar{v}} \times \prod_{\dot{E}} M_{\bullet} & \longrightarrow & \prod_{\dot{E}} M_{\bullet} \times M_{\bullet} & \longrightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ C^0 & \longrightarrow & \prod_{c \in \pi_0(V, E)} M_{t(c)} \times \prod_E M_v \times M_{\bar{v}} & \xrightarrow{\partial} & \prod_{\dot{E}} M_{\bullet} & \longrightarrow & 0 \end{array}$$

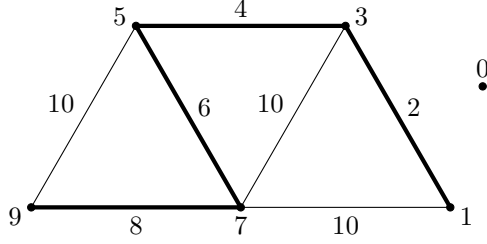


FIGURE 2. Labels indicate a specific discrete Morse function [19, 2.1] associated with the collapse of the bold subtree. Each vertex with Morse value less than 10 can be paired with its unique critical edge (of lesser Morse value) to iteratively collapse the bold subtree. This corresponds to the fact that only 0, 1 and 10 are critical values (cf. [19, 2.6]).

That is, each of the remaining  $\bullet \subseteq \check{e}$  coordinates in  $C^1$  for all  $\check{e} \in \check{E}$  is paired with one of the two remaining  $\bullet \subseteq v \subseteq \check{e}$  coordinates in  $C^2$ .

It remains to describe  $\partial$  and its cokernel in relation to  $\mathbf{S}_k$ . Set  $E_k$  to be the maximal elements of  $\mathbf{S}_k$  and note that the restriction of  $\partial$  to  $\prod_{c \in \pi_0(V, E)} M_{t(c)}$  is zero. Hence, we may consider the following diagram

$$\begin{array}{ccccc}
 \prod_{c \in \pi_0(V, E)} M_{t(c)} \times \prod_E M_v \times M_{\bar{v}} & \longrightarrow & \prod_E M_v \times M_{\bar{v}} & \longrightarrow & \prod_{E_k} M_v \times M_{\bar{v}} \\
 & \searrow \partial & \downarrow \partial_E & & \downarrow \partial_k \\
 & & \prod_{\check{E}} M_{\bullet} & \longrightarrow & \prod_{\{e_k\}} M_{\bullet}
 \end{array}$$

where the unlabeled homomorphisms are the canonical projections,  $\partial_E$  is defined to make the triangle commute, and  $\partial_k$  is associated to  $\mathbf{S}_k$  with  $\gamma_k$  chosen as a spanning tree. The above square also commutes because the restriction of  $\partial_E$  to the  $E - E_k$  coordinates is zero in the  $e_k$  coordinate. Roughly speaking, this corresponds to the fact that edges in  $E - E_k$  are not incident to the 2-cell corresponding to  $\mathbf{S}_k$ .

In this way, we have identified  $\text{coker}(\partial_E)$  with  $\lim_{\mathbf{S}}^2 M$  and  $\text{coker}(\partial_k)$  with  $\lim_{\mathbf{S}_k}^2 M$ . Moreover,  $\{\partial_k\}_{1 \leq k \leq m}$  determine  $\partial_E$  and with these identifications,  $\eta^*$  corresponds to

$$\left( \prod_{\check{E}} M_{\bullet} \right) / \text{Im}(\partial_E) \rightarrow \left( \prod_{\check{E}} M_{\bullet} \right) / \left( \prod_{\check{E}} \text{Im}(\partial_k) \right)$$

so that

$$(38) \quad \ker(\eta^*) \cong \left( \prod_{\check{E}} \text{Im}(\partial_k) \right) / \text{Im}(\partial_E).$$

In particular, (37) follows and the proof is complete.  $\square$

**Remark 7.5.** As previously noted, Lemma 7.4 depends only of the shape of  $\bar{\mathbf{S}}$ . For instance, Figure 4 applies to any  $D: \mathbf{P}^{op} \rightarrow R\text{-mod}$ . If

$$\text{Im}(\partial_E) = \prod_{\{e_1, e_2, e_3\}} \text{Im}(\partial_k) = (\langle D_1, D_2, D_3 \rangle, \langle D_2, D_3, D_4 \rangle, \langle D_2, D_4, D_5 \rangle).$$

where  $D_i = D_{\{i\}}$  as above, then  $\lim_{\mathbf{P}}^2 D \cong \lim_{\mathbf{S}_1}^2 D \times \lim_{\mathbf{S}_2}^2 D \times \lim_{\mathbf{S}_3}^2 D$ .

Notice also that the poset in the proof of Lemma 7.4 does not have to be finite and this proof applies to group cohomology calculations for infinitely generated Coxeter groups which are 2-spherical (cf. Proposition 6.1).  $\square$

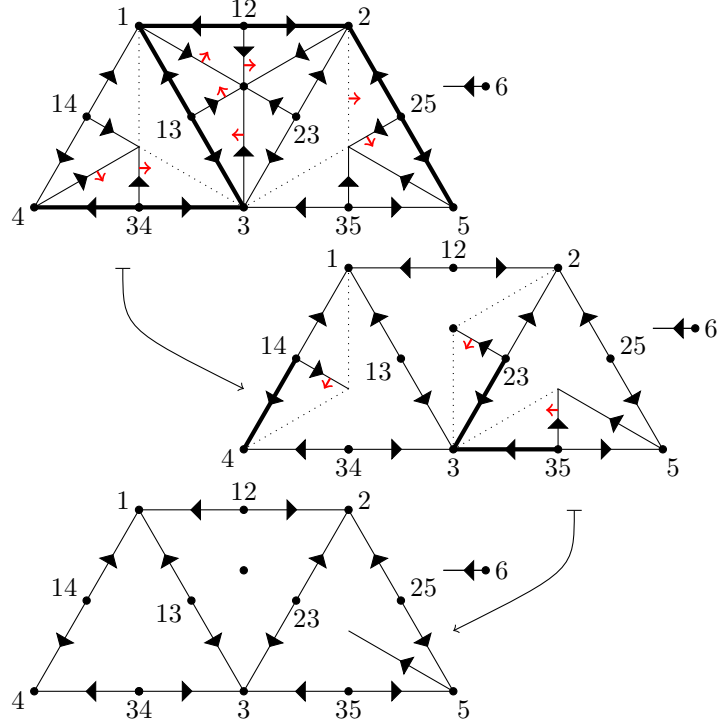


FIGURE 3. A picture of two successive (algebraic) discrete Morse theory pairings. The top figure is the Hasse diagram for a finite poset  $\mathbf{P}$  isomorphic to  $\bar{\mathbf{S}}$  covered by Lemma 7.4. The first pairing is associated to the discrete Morse theory pairing in Figure 2 and the second pairing is a further simplification.

**Example 7.6.** Let us also recall  $K := K(A)$  from Example 5.4. Here  $H^*(BK, \mathbb{Q})$  is not concentrated in odd degrees, but

$$H^*(BT, \mathbb{Q})^W \cong \lim_{\mathbf{S}}^0 H^*(BK_I, \mathbb{Q})$$

is known to be isomorphic to  $\mathbb{Q}[c_2, y_4, y_6]$  additively and Theorem 6.9 gives that  $\lim_{\mathbf{S}}^2 H^*(BK_I, \mathbb{Q}) = 0$ . Hence, an Euler characteristic calculation implies that  $\lim_{\mathbf{S}}^1 H^*(BK_I, \mathbb{Q})$  has Poincaré series given by:

$$\begin{aligned} \frac{3}{(1-t^2)^3(1-t^4)} &= \frac{3}{(1-t^2)^2(1-t^4)(1-t^6)} - \frac{1}{(1-t^2)^4} - \frac{1}{(1-t^2)(1-t^4)(1-t^6)} \\ &= \frac{t^4}{(1-t^2)(1-t^4)(1-t^6)}. \end{aligned}$$

In particular,  $H^5(BK, \mathbb{Q}) \cong \lim_{\mathbf{S}}^1 H^4(BK_I, \mathbb{Q})$  has rank one and additively  $H^*(BK, \mathbb{Q})$  agrees with  $\mathbb{Q}[c_2, y_4, y_6] \otimes \Lambda(x_5)$ . In fact, it is possible to construct a weak equivalence

$$(39) \quad BK_{\mathbb{Q}}^{\wedge} \simeq K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 5) \times K(\mathbb{Q}, 6);$$

see Example A.7 below.  $\square$

## 8. INVARIANT THEORY AND THE PROOF OF THEOREM 6.9

The main aim of this section is to prove Theorem 6.9 so that, with Theorem 6.8, Theorem C is proved. Our proof of Theorem 6.9 will use invariant theory arguments to show that the left and right terms of (37) vanish for  $M = H^n(D_K, \mathbb{Q})$  with  $K$

2-spherical. Our primary reference for invariant theory is [46], but we caution the reader that we will need to consider infinite reflection groups.

**Definition 8.1.** An involution  $r \in GL_n(\mathbb{Q})$  is a reflection if its fixed space  $H_r$  is  $n - 1$  dimensional.

For each reflection  $r$ , there is a linear form  $l_r: \mathbb{Q}^n \rightarrow \mathbb{Q}$  is determined, up to scalar, by the fact  $l_r$  has kernel  $H_r$ .

Let us recall how the Weyl group  $W$  of a Kac–Moody group are reflections groups. The action  $W$  on the the Lie algebra  $\mathfrak{h}$  of the torus is generated by  $n$  reflections

$$s_i(h) = h - \alpha_i(h)\alpha_i^\vee$$

for all  $h \in \mathfrak{h}$  with  $\alpha_i \in \mathfrak{h}^*$  a simple root,  $\alpha_i^\vee \in \mathfrak{h}$  a corresponding simple coroot and  $n$  the size of the generalized Cartan matrix. In particular, the simple roots [35, 1.2, 1.3] give canonical linear forms  $l_{s_i}$  and these sets of simple roots and coroots are part construction of the underlying Kac–Moody Lie algebra [35, 1.1–3]. For any subset  $I \subseteq \{1, \dots, n\}$  the subgroup  $W_I := \langle s_i | i \in I \rangle$  are the Weyl groups for corresponding parabolic subgroups  $K_I$  of  $K$  which—when  $W_I$  is finite—appear as  $D_K(I) := BK_I$  in (2).

Due to the following proposition, we adopt the standing convention that each  $t_i$  in  $\mathbb{Q}[t_1, \dots, t_r]$  has degree 2.

**Proposition 8.2.** *The subgroup  $W_I$  of the Weyl group of a Kac–Moody group is finite if and only if its collection of reflecting hyperplanes is finite. For any finite  $W_I$ , there is natural isomorphism  $H^*(BK_I, \mathbb{Q}) \cong \mathbb{Q}[t_1, \dots, t_r]^{W_I}$  so that  $H^*(D_K, \mathbb{Q})$  and  $\mathbb{Q}[t_1, \dots, t_r]^{W_I}$  are naturally isomorphic functors.*

**Proof:** For each reflection  $r$  in  $W$  the associated  $l_r$  is the root associated to  $r$ . In particular, none of these real roots are scalar multiples of any other [35, 1.2, 1.3] and thus no two  $H_r$  coincide. That is, there is a one-to-one correspondence  $r \leftrightarrow H_r$ .

The second statement is contained in Theorem 6.1. In particular, the action of  $W_I$  on  $H^2(BT, \mathbb{Q})$  is dual to the action of  $W_I$  on the Lie algebra  $\mathfrak{h}$  of  $T$  and it is well known that the rational cohomology of the classifying space of a compact connected Lie group is given by the Weyl group invariants of  $H^*(BT, \mathbb{Q})$  (see, e.g., [31]).  $\square$

**Definition 8.3.** For a linear group  $H \leq GL(V)$ , the relative invariants of  $H$  (with respect to the determinant)  $\mathbb{Q}[V]_{\det}^H$  is the set of  $f \in \mathbb{Q}[V]$  such that  $hf = \det(h)f$  for all  $h \in H$ .

The relative invariants of Weyl groups of parabolic subgroups of Kac–Moody group will play a key role in our calculations.

**Proposition 8.4.** *If  $R \leq GL_n(\mathbb{Q})$  is generated by reflections such that the collection of reflecting hyperplanes is infinite, then  $\mathbb{Q}[t_1, \dots, t_n]_{\det}^R = 0$ .*

**Proof:** Recall that if  $h \in \mathbb{Q}[t_1, \dots, t_n]_{\det}^s$  for some reflection  $s$ , then that  $l_s$  divides  $h$  [46, p.226]. Hence, if the number of distinct reflecting hyperplanes is at least  $d$ , then the degree of  $h$  is a least  $d$ .  $\square$

**Proposition 8.5.** *If  $R \leq GL_n(\mathbb{Q})$  is generated by two reflections  $s \neq t$  and finite, then*

$$\mathbb{Q}[t_1, \dots, t_n] = \mathbb{Q}[t_1, \dots, t_n]_{\det}^R \oplus \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^s, \mathbb{Q}[t_1, \dots, t_n]^t)$$

as a graded vector space.

**Proof:** Here  $|t_i| = 2$  and  $R$  is a dihedral group. It is easy to see that

$$(40) \quad \mathbb{Q}[t_1, \dots, t_n]_{\det}^R \cap \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^s, \mathbb{Q}[t_1, \dots, t_n]^t) = 0.$$

(cf. [1, pp.479–480]). Hence, the internal direct sum

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^R \oplus \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^s, \mathbb{Q}[t_1, \dots, t_n]^t)$$

is  $\mathbb{Q}[t_1, \dots, t_n]$  if and only if these graded vector spaces have the same dimension in each degree. By [46, p.227] the Poincaré series for  $\mathbb{Q}[t_1, \dots, t_n]_{\det}^R$  is

$$t^{|R|} \cdot P(\mathbb{Q}[t_1, \dots, t_n]^R, t)$$

where  $P(M^*, t)$  denotes the Poincaré series of a graded algebra  $M^*$ . For any integer  $k$  we have an identity of rational functions

$$\frac{1}{(1-x^2)(1-x^k)} - \frac{x^k}{(1-x^2)(1-x^k)} = \frac{1}{(1-x^2)} = \frac{2}{(1-x^2)(1-x)} - \frac{1}{(1-x)^2}.$$

Considering the Poincaré series for  $\mathbb{Q}[t_1, \dots, t_n]^s$ ,  $\mathbb{Q}[t_1, \dots, t_n]^t$  and  $\mathbb{Q}[t_1, \dots, t_n]^R$  (see, e.g., [46, 1.3, 7.4]), the result follows from the above equality (scaled by  $\frac{1}{(1-x)^{n-2}}$ ) with  $x = t^2$  and  $k = \frac{|R|}{2}$ .  $\square$

We now summarize the invariant theory input for proving Theorem 6.9.

**Lemma 8.6.** *Let  $R \leq GL_n(\mathbb{Q})$  be generated by three distinct reflections  $s, t$ , and  $r$  such that  $\langle s, t \rangle$  and  $\langle t, r \rangle$  are finite with  $\langle s, t \rangle \cap \langle t, r \rangle = \langle t \rangle$ . If the collection of reflecting hyperplanes in  $R$  is infinite, then*

$$(41) \quad \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle s, t \rangle} \leq \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^r)$$

so that

$$(42) \quad \mathbb{Q}[t_1, \dots, t_n] = \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^s, \mathbb{Q}[t_1, \dots, t_n]^t, \mathbb{Q}[t_1, \dots, t_n]^r)$$

in each degree. More generally, if  $R \leq GL_n(\mathbb{Q})$  is generated by distinct reflections  $r_1, \dots, r_l, t$  for  $l \geq 2$  such that each  $\langle r_i, t \rangle$  is finite and for each pair  $i \neq j$ ,  $1 \leq i, j \leq l$  the collection of reflecting hyperplanes in  $\langle t, r_i, r_j \rangle$  is infinite and  $\langle r_i, t \rangle \cap \langle t, r_j \rangle = \langle t \rangle$ , then

$$(43) \quad \bigoplus_{1 \leq k \leq l, k \neq i} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_k, t \rangle} \leq \mathbb{Q}[t_1, \dots, t_n]^{r_i}$$

where  $\oplus$  denotes internal direct sum.

**Proof:** By Proposition 8.5,  $\mathbb{Q}[t_1, \dots, t_n]$  coincides with

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle s, t \rangle} \oplus \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^s, \mathbb{Q}[t_1, \dots, t_n]^t)$$

in each degree and, in particular, induces a splitting of  $\text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^r)$ . Considering the splitting

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r, t \rangle} \oplus \text{span}_{\mathbb{Q}}(\mathbb{Q}[t_1, \dots, t_n]^r, \mathbb{Q}[t_1, \dots, t_n]^t)$$

(41) is equivalent to

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle s, t \rangle} \cap \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r, t \rangle} = \mathbb{Q}[t_1, \dots, t_n]_{\det}^R = 0$$

which follows from Proposition 8.4. Thus, we have (41) and (42) is immediate.

We prove (43) by induction on  $l$  and have just discussed the  $l = 2$  base case. Now assume that for  $i \neq j$ ,  $1 \leq i, j \leq l$  we have

$$(44) \quad \bigoplus_{1 \leq k \leq l, j \neq k \neq i} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_k, t \rangle} \leq \mathbb{Q}[t_1, \dots, t_n]^{r_i}.$$

By the base case, we have

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_j, t \rangle} + \bigoplus_{1 \leq k \leq l, j \neq k \neq i} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_k, t \rangle} \leq \mathbb{Q}[t_1, \dots, t_n]^{r_i}$$

for any  $1 \leq i \leq l$ . To see that

$$(45) \quad \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_j, t \rangle} \cap \bigoplus_{1 \leq k \leq l, j \neq k \neq i} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_k, t \rangle} = 0$$

as graded vector spaces, we note that Propositions 8.4 and 8.5 together imply

$$\mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle r_j, t \rangle} \cap \mathbb{Q}[t_1, \dots, t_n]^{r_j} = 0$$

so that (45) follows from (44) with  $i$  and  $j$  interchanged.  $\square$

Recall that Example 7.3 describes how the somewhat technical looking statement of Lemma 8.6 applies to a concrete situation. Notice also that in Example 7.3 only (42) is needed to obtain the desired vanishing. We will use (43) to handle cases where  $|E_S| \gg |V_S|$  for the graph  $(V_S, E_S)$ . Lemmas 7.2, 7.4 and 8.6 now combine to prove Theorem 6.9.

**Proof of Theorem 6.9:** By Proposition 8.2,  $\lim_{\mathbf{S}}^2 H^n(D_K, \mathbb{Q}) = 0$  for all  $n \geq 0$  if and only if  $\lim_{\mathbf{S}}^2 \mathbb{Q}[t_1, \dots, t_r]^{W_I} = 0$ . For  $K$  a 2-spherical Kac–Moody group, fix a spanning forest for  $(V_S, E_S)$ . Construct  $\mathbf{S}_k$  as in Lemma 7.4 to obtain (37) for the functor  $M$  defined via  $I \mapsto \mathbb{Q}[t_1, \dots, t_r]^{W_I}$ .

Thus,  $\lim^2 H^n(D_K, \mathbb{Q}) = 0$  if and only if

$$(46) \quad \text{Im}(\partial_E) = \prod_{\dot{E}} \text{Im}(\partial_k) = \prod_{\dot{E}} M_{\emptyset}.$$

where here and below we use the definitions from the the proof of Lemma 7.4 freely.

Set  $I_k = \bigcup_{I \in \mathbf{S}_k} I$  and note that  $W_{I_k} := \langle s_i | i \in I_k \rangle$  is infinite since  $|I_k| = n_k \geq 3$  and  $K$  is 2-spherical. Indeed, Lemma 8.6 applies to some  $W_I$  for  $I \subseteq I_k$  with  $|I| = 3$  so that the right equality of (46) now follows easily from (35) and (42) given in Lemmas 7.2 and 8.6, respectively. Define  $V' \subseteq V_S$  to be the set of vertices not incident with some edge in  $\dot{E}$  and note

$$(47) \quad \begin{aligned} \prod_{\dot{E}} M_{\bullet} / \text{Im}(\partial_E) &\cong \left( \prod_{e_k \in \dot{E}} M_{\bullet} / \partial_E(M_{v_k} + M_{\bar{v}_k}) \right) / \left( \text{Im}(\partial_E) / \prod_{e_k \in \dot{E}} \partial_E(M_{v_k} + M_{\bar{v}_k}) \right) \\ &\cong \prod_{I \in \dot{E}} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{W_I} / \partial_E \left( \prod_{e \supseteq \{i\} \in V'} \mathbb{Q}[t_1, \dots, t_n]^{s_i} \right) \end{aligned}$$

where the bottom right product is taken over pairs in  $\{(e, v) | v \subseteq e\} \subseteq E \times V'$  and  $s_i$  generates the Weyl group of  $K_{v=\{i\}}$ .

Now let  $v$  be an arbitrary leaf in the spanning forest, i.e.  $v$  is in the forest and adjacent to a unique vertex in the forest. In other words,  $v$  is incident with some edge in  $\dot{E}$  or  $v$  is adjacent to a unique vertex in  $(V_S, E_S)$ . Define  $L_v := \{v_0, \dots, v_{|L_v|-1}\}$  to be vertices adjacent to  $v$  where  $v_0$  is the unique vertex in the forest adjacent to  $v$ . See Figure 4.

By design, Lemma 8.6 implies

$$(48) \quad \bigoplus_{L_v - \{v_0\}} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{\langle s_{v_j}, s_v \rangle} \leq \mathbb{Q}[t_1, \dots, t_n]^{s_{v_0}}$$

In particular, we can partition  $\dot{E} = \prod \dot{E}_v$  into parts  $\dot{E}_v$  such that all edges in  $\dot{E}_v$  adjacent to a leaf  $v$ . Since  $v_0 \in V'$ , the expression (48) implies that  $\partial_E$  maps  $\mathbb{Q}[t_1, \dots, t_n]^{s_{v_0}}$  onto  $\prod_{I \in \dot{E}_v} \mathbb{Q}[t_1, \dots, t_n]_{\det}^{W_I}$ . Hence  $\partial_E$  is surjective so that (47)



vanishes as desired, the left hand equality of (46) holds, and the proof is complete.  $\square$

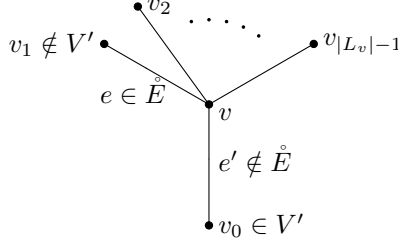


FIGURE 4. A local picture of  $(V_S, E_S)$  around some leaf  $v$  in the spanning forest.

As noted before the proof of Theorem 6.9, the expression (43) is particularly strong; for instance, it is straightforward to exhibit 2-spherical Kac–Moody groups so that the associated graph  $(V_S, E_S)$  is complete on  $k$  vertices for any  $k \in \{3, 4, 5, \dots\}$ . There is little reason to expect  $\lim_{\mathbf{S}}^2 H^*(W_I, \mathbb{Q})$  to vanish for a general Coxeter group  $W \leq GL_n(\mathbb{Q})$  whose abstract reflections as a Coxeter group are concrete reflections in the sense of Definition 8.4. However, one may wonder what analogs of (43) exist for such  $W$  whose abstract hyperplanes (represented, e.g., by walls in the Coxeter complex [12]) are in bijection with the concrete hyperplanes  $H_r$ . For the Weyl group of a Kac–Moody group, we pose the following question.

**Question 8.7.** For which Kac–Moody groups  $K$  does  $\lim^2 H^*(D_K, \mathbb{Q})$  vanish? More generally, when does  $\lim^{2n} H^*(D_K, \mathbb{Q}) = 0$  for  $n \geq 1$ ?

Notice that for any essentially 3-spherical  $K$  a positive answer to the first question implies a positive answer to the second. Notice also that if  $\lim^{2n} H^*(D_K, \mathbb{Q})$  vanishes for all  $n \geq 1$ , then all nilpotent elements of  $H^*(BK, \mathbb{Q})$  appear in odd degrees so that all products of nilpotent elements of  $H^*(BK, \mathbb{Q})$  vanish and the  $H^*(BT_K, \mathbb{Q})^W$ -module structure of  $H^*(BK, \mathbb{Q})$  determines  $H^*(BK, \mathbb{Q})$  as a ring (cf. Example A.7 below). The construction of examples which are not essentially 3-spherical begins with  $n \times n$  generalized Cartan matrices with  $n \geq 5$ .

#### APPENDIX A. RATIONAL PRODUCT DECOMPOSITIONS AND KAC-MOODY GROUPS ASSOCIATED TO DERIVED LIE ALGEBRAS

As stated in the introduction, this paper studies connected unitary Kac–Moody groups as constructed in [35, 7.4]. However, it is also common to consider (simply connected) unitary Kac–Moody groups  $K'(A)$  associated to derived Kac–Moody Lie algebras (see, e.g. [29, §2.5]). Some of our references—notably [30, 32, 48]—follow this practice. This appendix will clarify how our stated results and examples apply to this alternative setting, how [30, 48] apply to our setting, and why the rationalization of both types of groups split as a product of  $K(\pi, n)$ ’s. Throughout this appendix,  $K$  will always denote a unitary Kac–Moody group as in [35, 7.4] with standard torus  $T$  while  $K'$  will always denote a simply connected unitary Kac–Moody group as in [29, §2.5] with standard torus  $T'$ .

For  $K$  associated to a Kac–Moody Lie algebra  $\mathfrak{g}$  such that  $K'$  is associated to the derived Kac–Moody Lie algebra  $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ , the inclusion  $K' \leq K$  induces a homeomorphism of  $CW$ -complexes  $K'/T' \cong K/T$  (cf. [33, Appendix], [32, 1.13]). In fact, there are corresponding inclusions  $K'_I \leq K_I$  of compact groups for all  $I \in \mathbf{S}$

inducing homeomorphisms  $K'/K'_I \cong K/K_I$  which give a homeomorphism of Tits buildings

$$\mathrm{hocolim}_{I \in \mathbf{S}} K'/K'_I \xrightarrow{\sim} \mathrm{hocolim}_{I \in \mathbf{S}} K/K_I$$

and the quotient  $\overline{T} := T/T'$  is a torus of rank equal to corank of the generalized Cartan matrix so that  $T \cong T' \times \overline{T}$  (cf. [37, 5.1]).

**Remark A.1.** Since each  $(BK')_p^\wedge$  can be constructed as in (1) (cf. [32, 4.2.4] noting [17, 2.9]) and (18) holds with respect to  $BT'$ , all the results of Sections 2-4 apply to the alternative setting. In particular, Theorem A holds with either or both of  $K$  and  $\Gamma$  replaced with some  $K'$  as in [29, §2.5].  $\square$

Because  $K$  is a finite-type [33, Appendix]  $H$ -space [31], its fundamental group is finitely generated abelian and its covering spaces are well-understood [31, I: §3]. Path lifting gives any cover of  $K$ , and in particular the universal cover  $\tilde{K}$ , the structure of a topological group [8]. Moreover, the map of fiber sequences

$$(49) \quad \begin{array}{ccc} T' & \xrightarrow{\quad} & T \\ \downarrow & & \downarrow \\ K' & \xrightarrow{\quad} & K \\ \downarrow & & \downarrow \\ K'/T' & \xrightarrow{\sim} & K/T \end{array}$$

yields the identification  $\pi_1(\overline{T}) \cong \pi_1(K)$ .

**Proposition A.2.** *With the preceding notation, any  $K$  or  $K'$  is rationally a countable product  $\prod_{n>0} K(V_n, n)$  with each  $V_n$  a finite dimensional rational vector space with basis  $J_n$ .*

**Proof:** Since  $K'$  is simply-connected and  $H^*(K', \mathbb{Q})$  is freely generated as a ring, the algebra generators induce a natural map  $K' \rightarrow \prod_{n>1} K(V_n, n)$  which induces a weak equivalence  $(K')_{\mathbb{Q}}^\wedge \simeq \prod_{n>1} K(V_n, n)$  [24].

By [31, I: §3-2], there is an equivalence  $\tilde{K} \times \overline{T} \simeq K$  so that  $K$  also has this form with  $|J_1| = \mathrm{rank}(\overline{T})$ .  $\square$

Of course, Proposition A.2 applies to any finite type  $H$ -space.

**Proposition A.3.** *With the preceding notation, if  $K'$  or  $K$  is indefinite, indecomposable, then  $J_{\mathrm{odd}} := \bigcup_{n=2i+1>0} J_n$  is described in Table 1.*

**Proof:** In the alternative setting, these results for  $K'$  appear in [30, 48]. In particular, an element in  $J_3$  corresponds to a  $W$ -invariant bilinear form.

For  $\tilde{K}$  the universal cover, we have a diagram of *homomorphisms of topological groups* [8]

$$\begin{array}{ccc} & & \tilde{K} \\ & \nearrow \exists! & \downarrow p \\ K' & \xrightarrow{\quad} & K \end{array}$$

TABLE 1.  $J_{\text{odd}}$  for indefinite, indecomposable  $K'$  and  $K$ 

	$K'$	$K$
symmetric	$ J_{\text{odd}}  =  J_3  = 1$	$ J_3  = 1,  J_1  = \text{rank}(\overline{T}) =  J_{\text{odd}}  - 1$
not symmetric	$J_{\text{odd}} = \emptyset$	$ J_{\text{odd}}  =  J_1  = \text{rank}(\overline{T})$

which induces  $K' \simeq_{\mathbb{Q}} \tilde{K}$  and lets us complete the table. More explicitly, (49) induces can map of short exact sequences of Hopf algebras [33, Theorem 2.2]

$$\begin{array}{ccccc}
 H^*(K/T, \mathbb{Q}) \otimes_S \mathbb{Q} & \longrightarrow & H^*(K, \mathbb{Q}) & \longrightarrow & \Lambda(y_1, \dots, y_{\text{rank}(\overline{T})}, x_1, \dots, x_k) \\
 \parallel & & \downarrow & & \downarrow \\
 H^*(K'/T', \mathbb{Q}) \otimes_S \mathbb{Q} & \longrightarrow & H^*(K', \mathbb{Q}) & \longrightarrow & \Lambda(x_1, \dots, x_k)
 \end{array}$$

by construction such that the degree of each  $y_i$  is one and  $\tilde{K} \times \overline{T} \simeq K$  implies  $K' \simeq_{\mathbb{Q}} \tilde{K}$ . In the terminology of [33], these  $y_i$  correspond to a basis for the degree two part of the generalized invariants  $\mathcal{I}$  which agrees with the degree two invariants  $H^2(BT, \mathbb{Q})^W$ .  $\square$

**Remark A.4.** Propositions A.2 and A.3 imply that all the results of Sections 5-6 apply to the alternative setting—including Theorem B—with either or both of  $K$  and  $\Gamma$  replaced with some  $K'$  as in [29, §2.5].  $\square$

**Theorem A.5.** *If  $K'$  is 2-spherical, then  $\lim^2 H^*(D_{K'}, \mathbb{Q}) = 0$  where  $D_{K'}$  is defined via  $I \mapsto K'_I$  for  $I \in \mathbf{S}$ .*

**Proof:** There is a canonical surjection  $\lim^2 H^*(D_K, \mathbb{Q}) \twoheadrightarrow \lim^2 H^*(D_{K'}, \mathbb{Q})$ . That is, each  $H^*(BT, \mathbb{Q})^{W_I}$  is naturally a free  $\mathbb{Q}[J_1]$ -module generated by  $H^*(BT', \mathbb{Q})^{W_I}$  for all  $I \in \mathbf{S}$  so that  $\mathbb{Q}[J_1]$  is the kernel of this surjection.  $\square$

**Remark A.6.** In this way, Theorem A.5 and Theorem 6.8 (noting Remark A.4) imply that Theorem C holds with  $K$  replaced by some  $K'$  as in [29, §2.5].  $\square$

We finish this appendix by describing how Example 5.4 changes when moving to the simply connected setting of [29]. Notice that Examples 7.1 and 7.3 are not effected. We also determine the rational type of  $BK(A)$ .

**Example A.7.** Now let  $K' := K'(A)$  be the simply-connected affine Kac-Moody group associated to the generalized Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then  $H^*(K', \mathbb{Q}) = \Lambda(y_3, y_5) \otimes \mathbb{Q}[x_4]$ . Here the degree one generator in  $H^*(K(A), \mathbb{Q})$  corresponds an element of  $H^2(BT, \mathbb{Q})^W$  and this generator does not appear in  $H^*(K'(A), \mathbb{Q})$ . With analogous definitions and Euler characteristic arguments,

$$(50) \quad \ker \left( V_{K'}^{T'} \rightarrow V_{K'}^{K'} \right) \cong H^5(BK', \hat{\mathbb{Q}}/\mathbb{Q}) \cong \hat{\mathbb{Q}}/\mathbb{Q}.$$

as in Example 7.6. Hence, each non-zero  $x \in \hat{\mathbb{Q}}/\mathbb{Q}$  is represented by a non-trivial  $f_x: BK' \rightarrow BK'$  with  $f_x B\iota$  is null and nontrivial  $f_x: BK' \rightarrow BG$  with  $G$  is compact connected Lie can be constructed as in Example 5.4.

Considering the bigraded algebra structure on cohomological spectral sequence [4, XII. 5.8] with

$$F_{\infty}^{i,k} = F_2^{i,k} = \lim_{I \in \mathbf{S}}^i H^k(BK'_I, \mathbb{Q})$$

the  $H^*(BT, \mathbb{Q})^W$ -module structure of  $H^*(BK'(A), \mathbb{Q})$  determines  $H^*(BK', \mathbb{Q})$  as a ring. Hence,  $H^*(BK', \mathbb{Q}) \cong \mathbb{Q}[x_4, x_6] \otimes \Lambda(y_5)$  as a ring so that—as in proof of Proposition A.2—we have a weak equivalence

$$(51) \quad (BK')_{\mathbb{Q}}^{\wedge} \simeq K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 5) \times K(\mathbb{Q}, 6).$$

Similarly,  $H^*(BK(A), \mathbb{Q}) \cong \mathbb{Q}[x_2, x_4, x_6] \otimes \Lambda(y_5)$  as a ring and (39) follows—as in proof of Proposition A.2.  $\square$

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